Graphical Models
Discrete Inference and Learning
Lecture 2

MVA

2019 – 2020

http://thoth.inrialpes.fr/~alahari/disinflearn

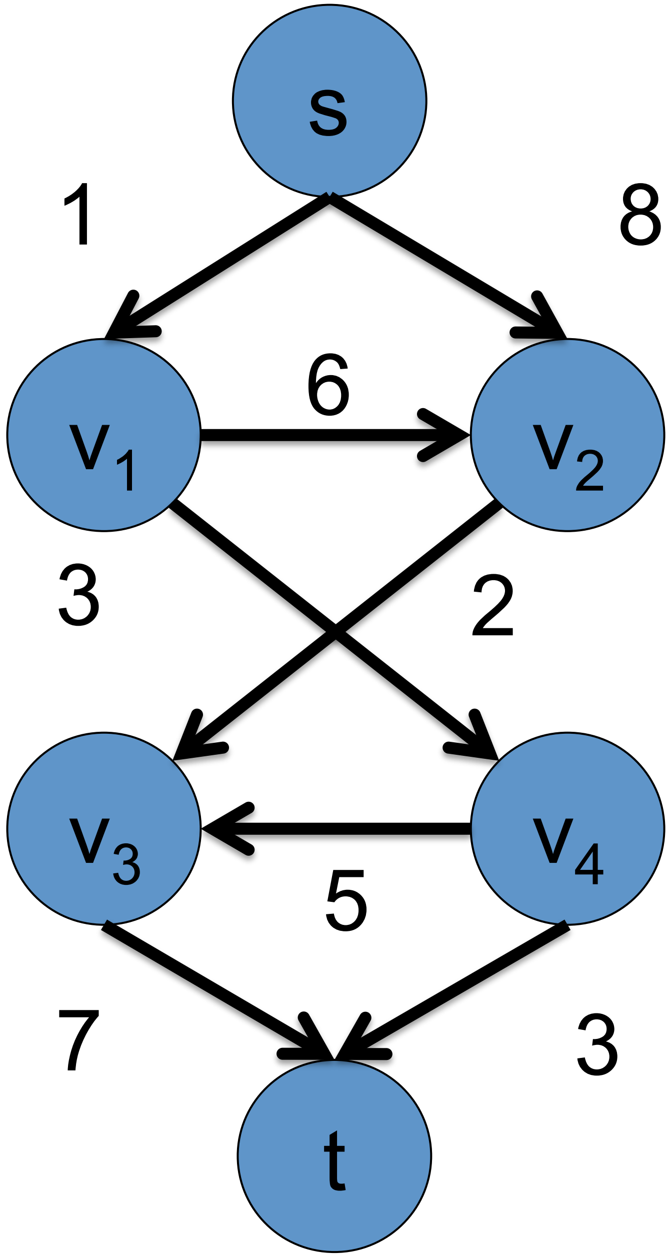
Slides based on material from M. Pawan Kumar

Outline

- Preliminaries
  - **s-t Flow**
  - s-t Cut
  - Flows vs. Cuts (details in next lecture)

- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

s-t Flow



Function flow: A ➔ R

Flow of arc ≤ arc capacity

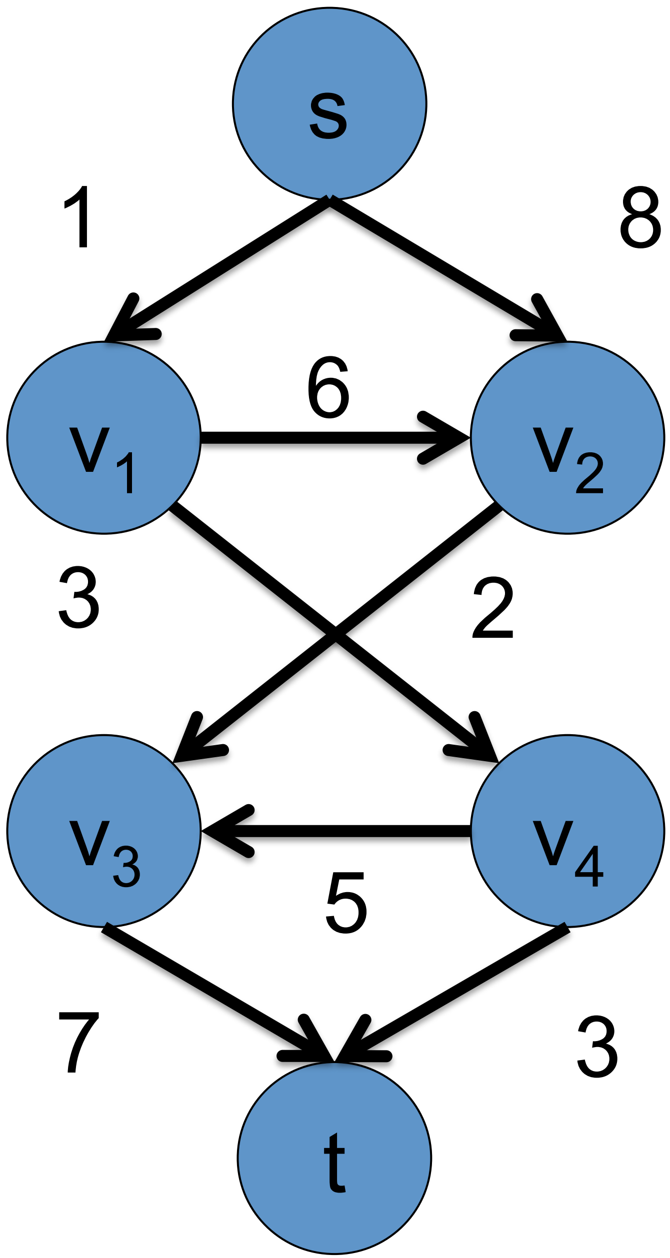
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

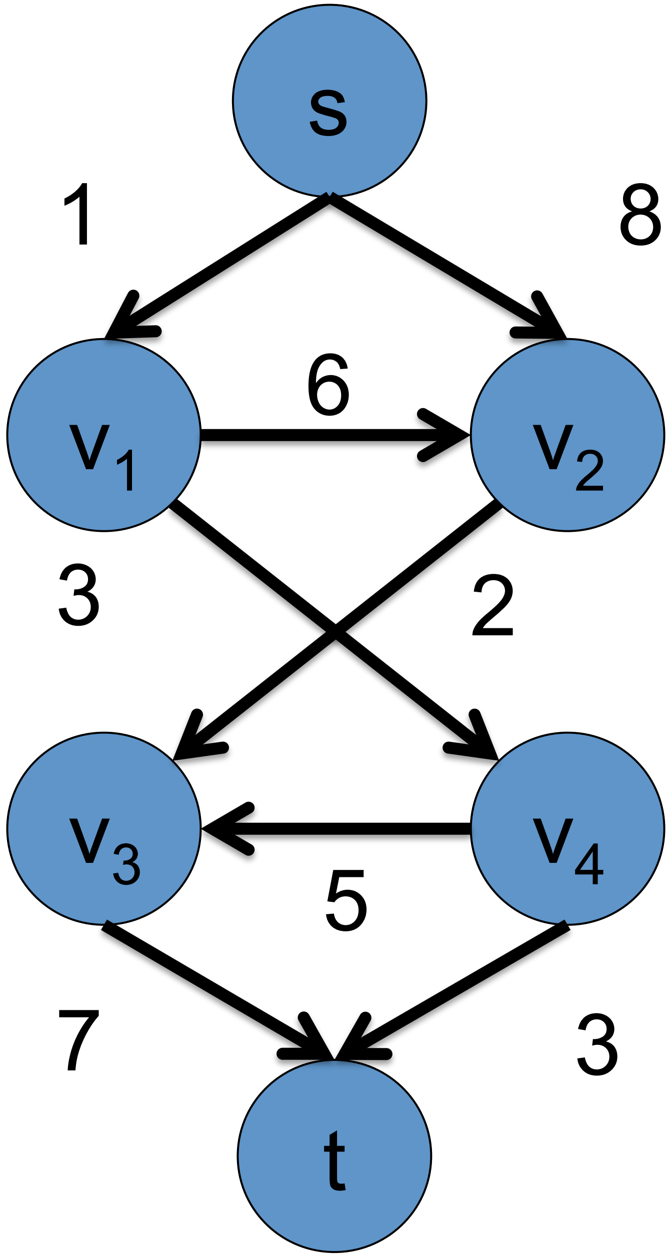
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A → R

flow(a) ≤ c(a)

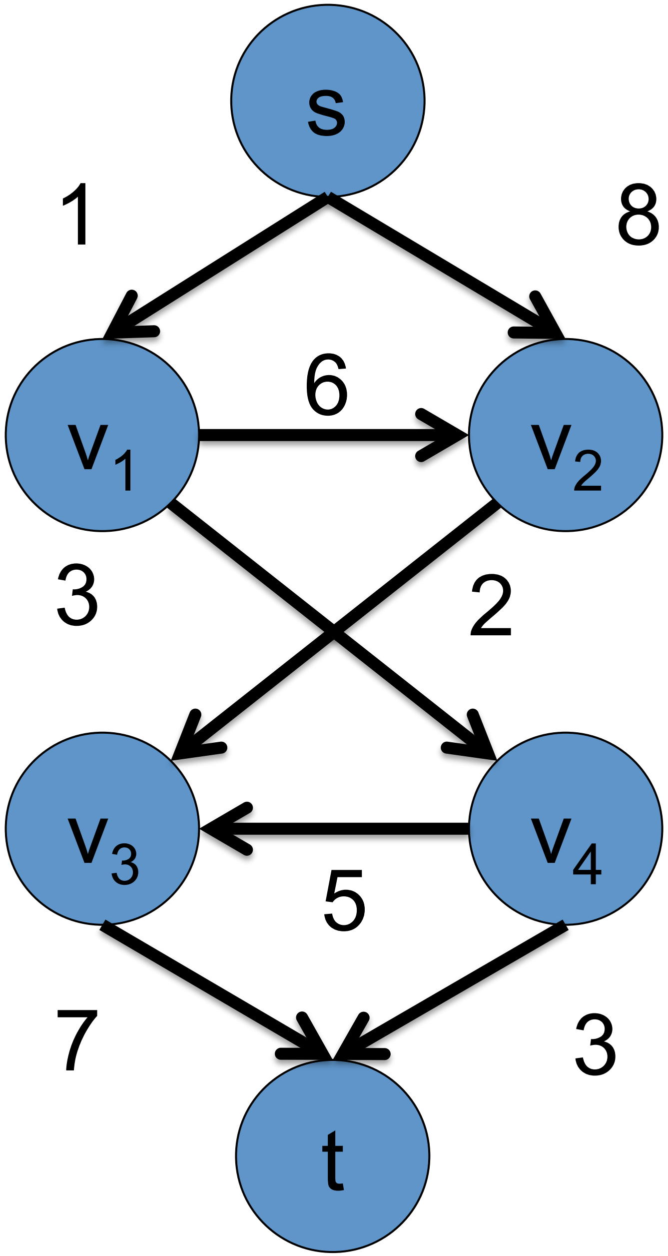
flow(a) ≥ 0

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

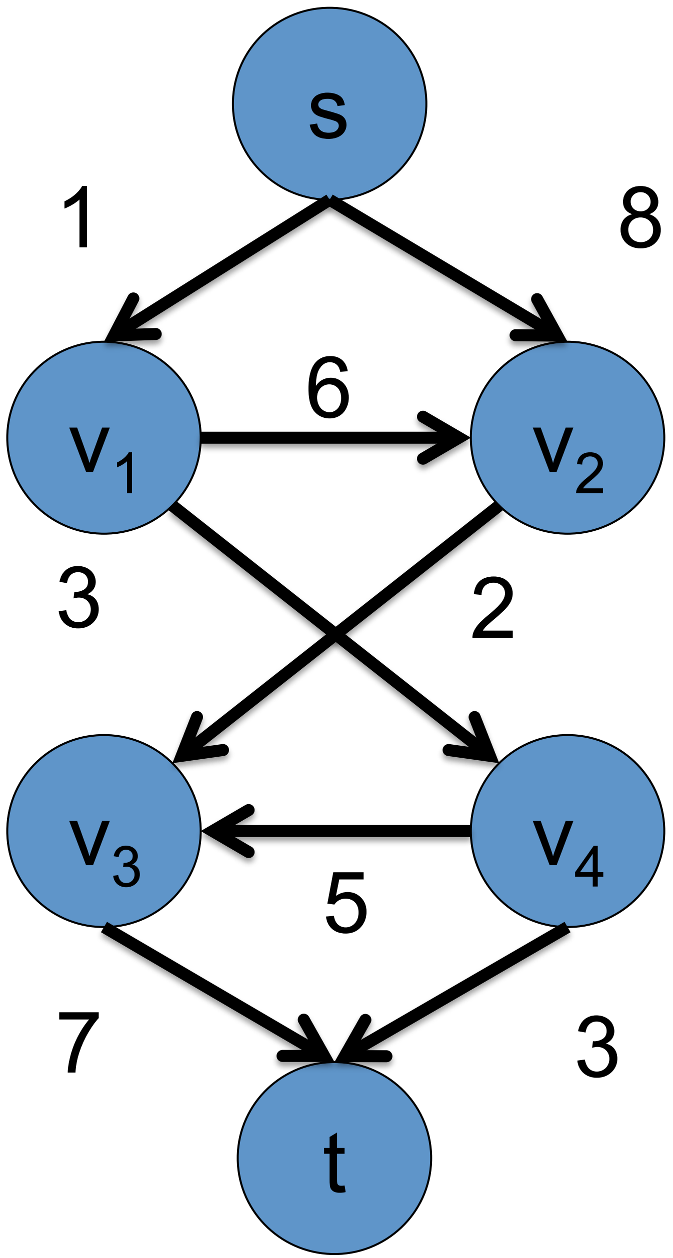
flow(a) ≥ 0

For all v ∈ V \ {s,t}

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

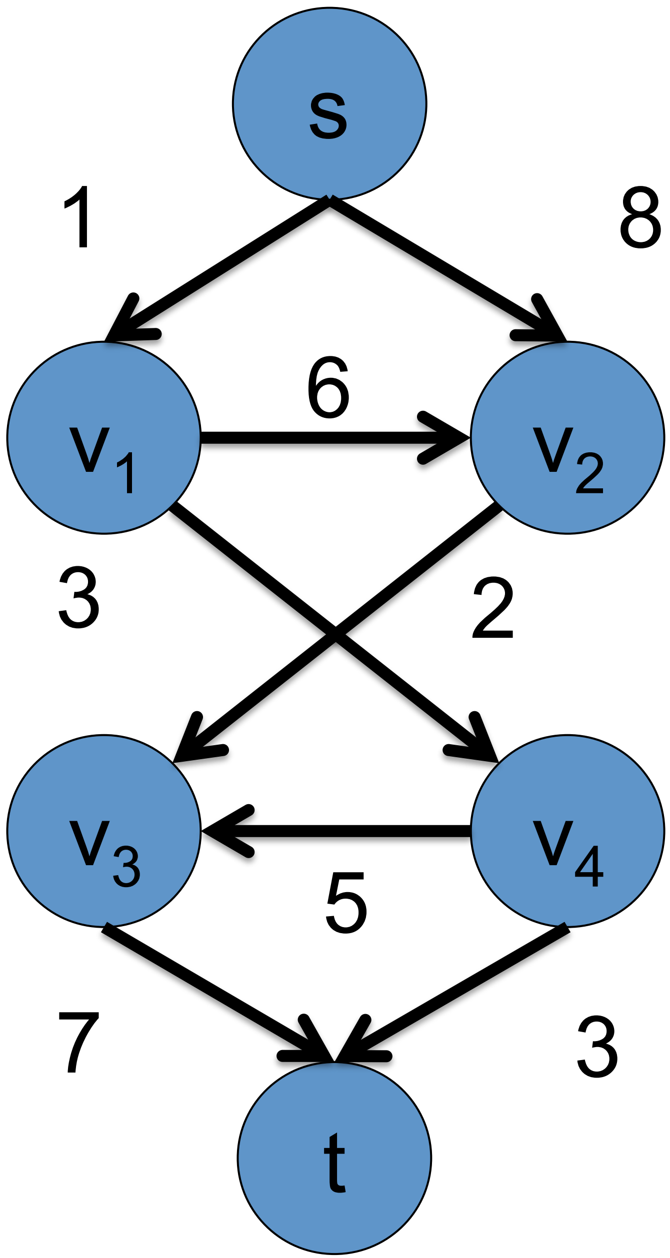
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$\Sigma_{(u,v)\in A}$ flow((u,v))

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

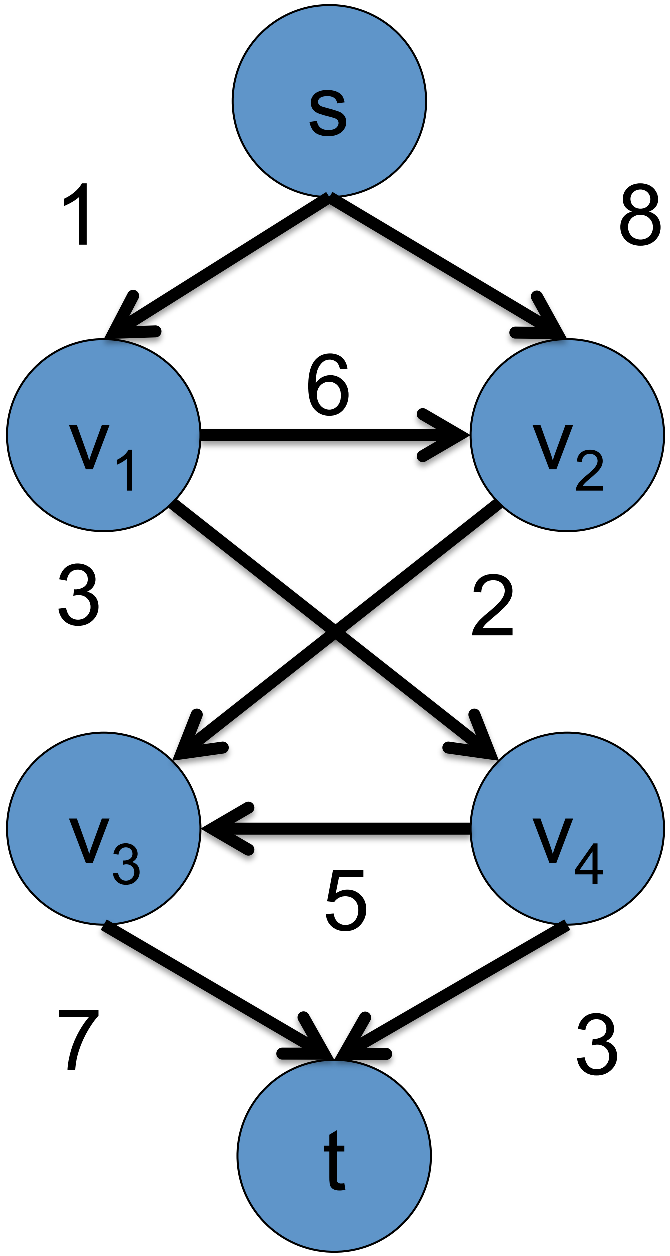
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$\Sigma_{(u,v)\in A}$ flow((u,v))

= $\Sigma_{(v,u)\in A}$ flow((v,u))

s-t Flow



Function flow: A ➔ R

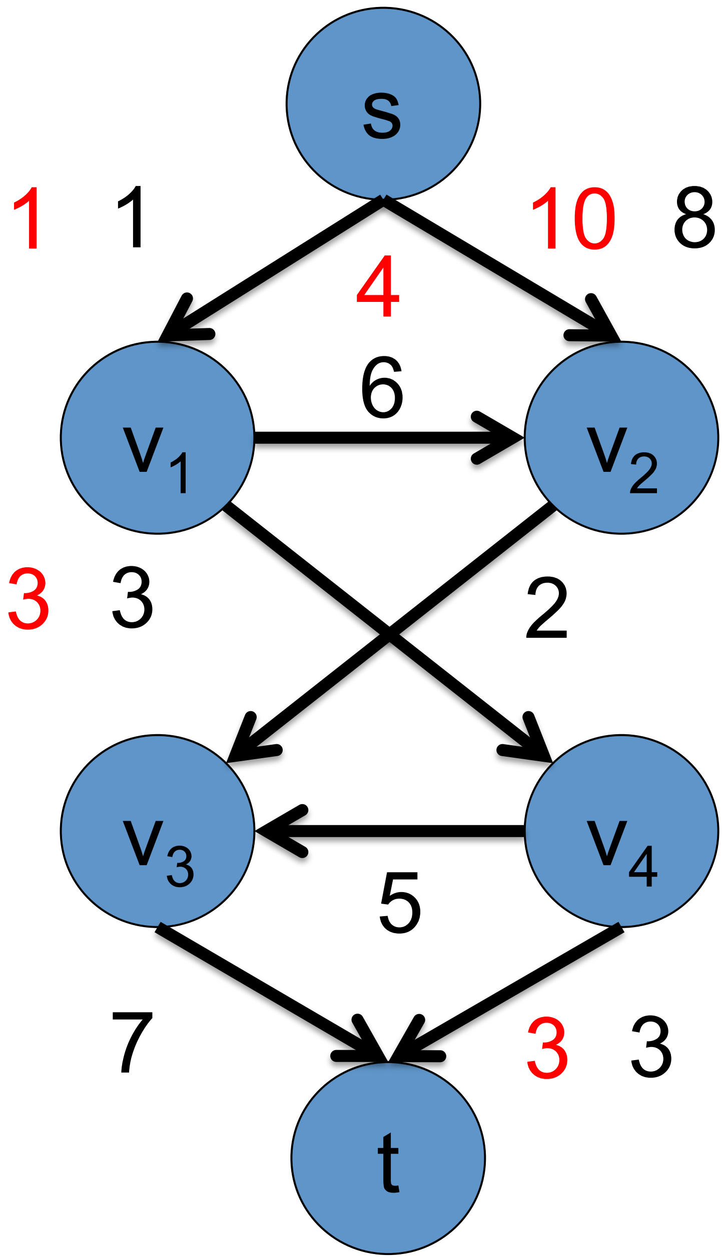
flow(a) ≤ c(a)

flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

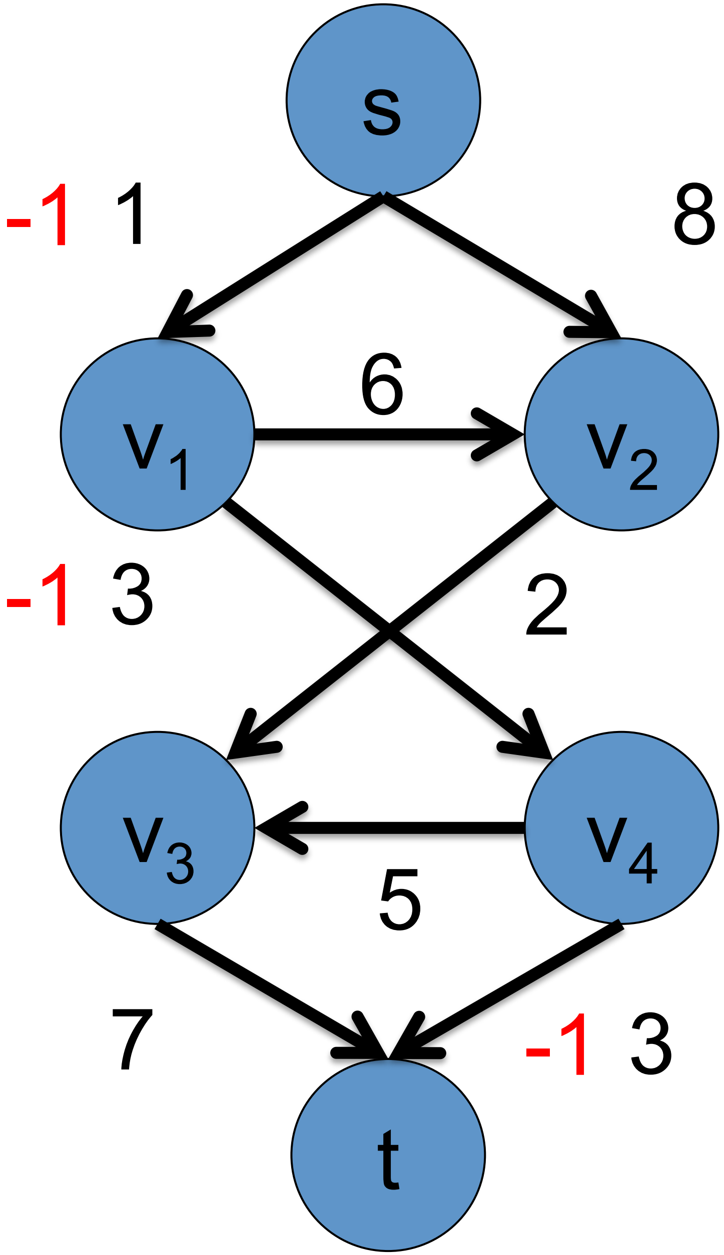
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

✗

s-t Flow



Function flow: A → R

flow(a) ≤ c(a)

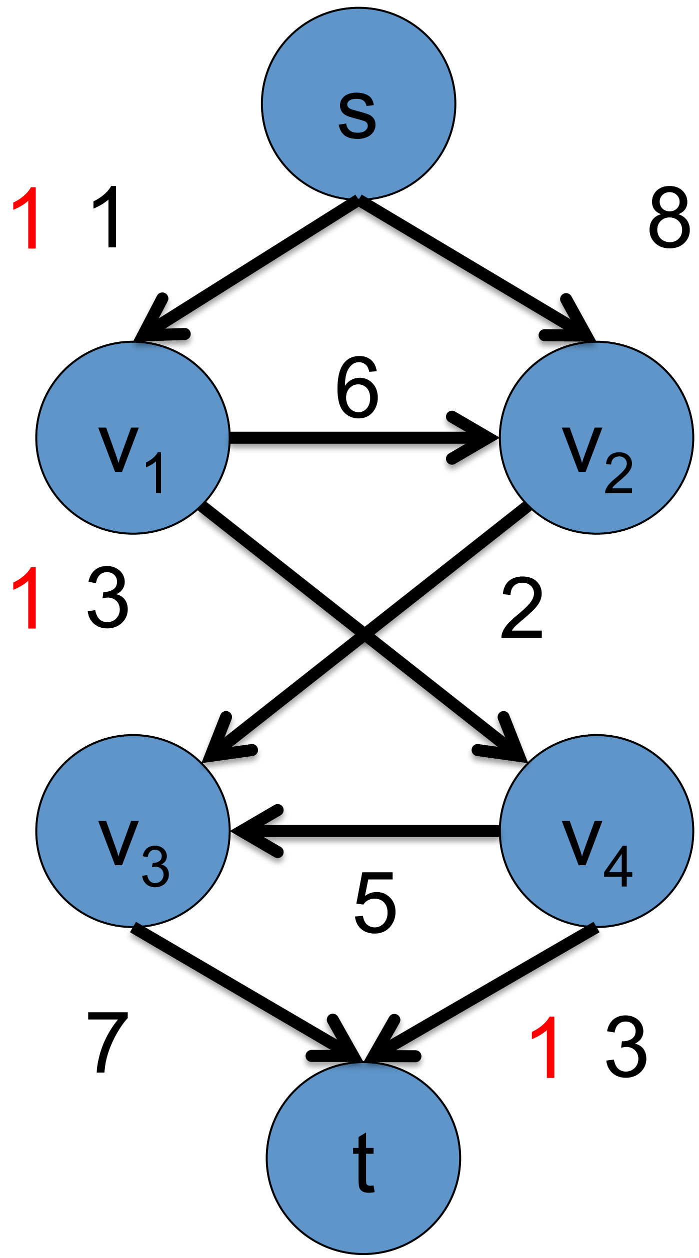
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

✗

s-t Flow



Function flow: A ➔ R

flow(a) $\leq$ c(a)

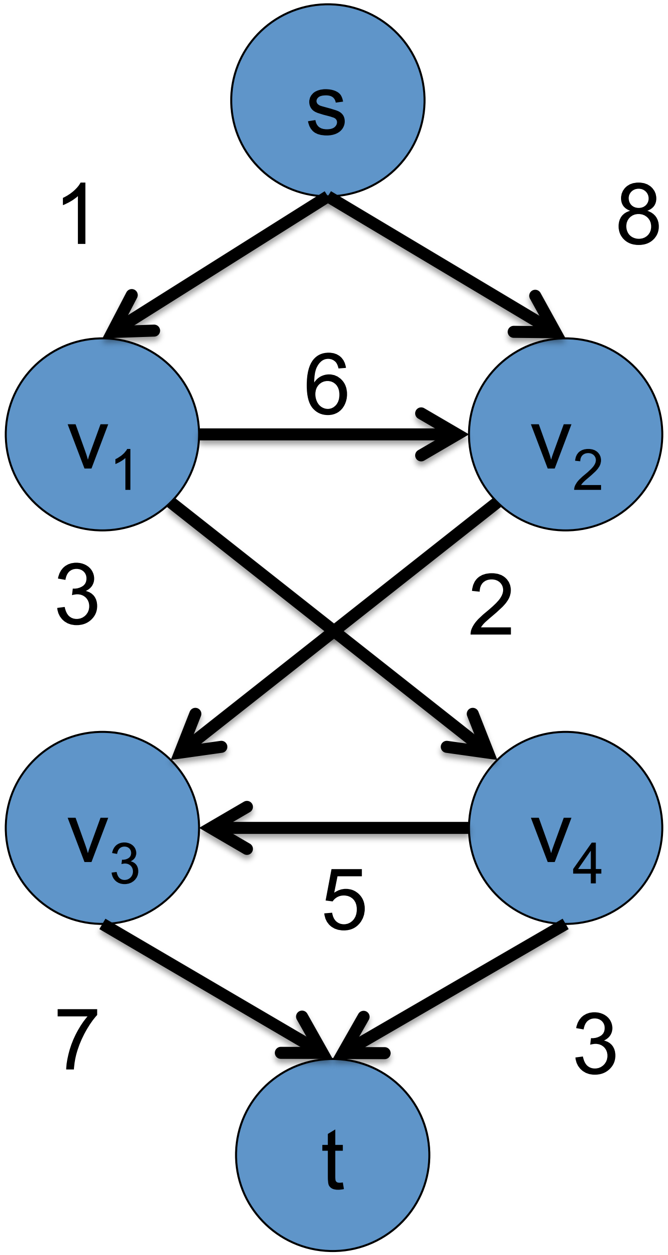
flow(a) $\geq$ 0

For all v $\in$ V \ {s,t}

$E_{flow}(v) = 0$

✓

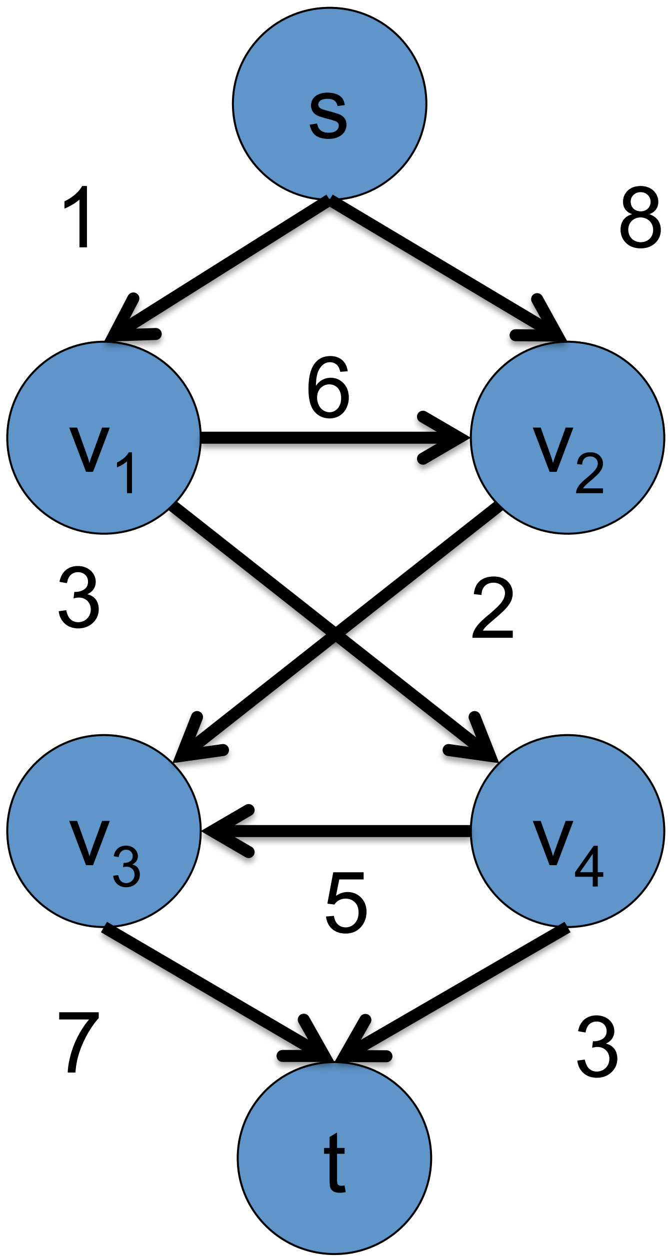
Value of s-t Flow



Outgoing flow of s

- Incoming flow of s

Value of s-t Flow

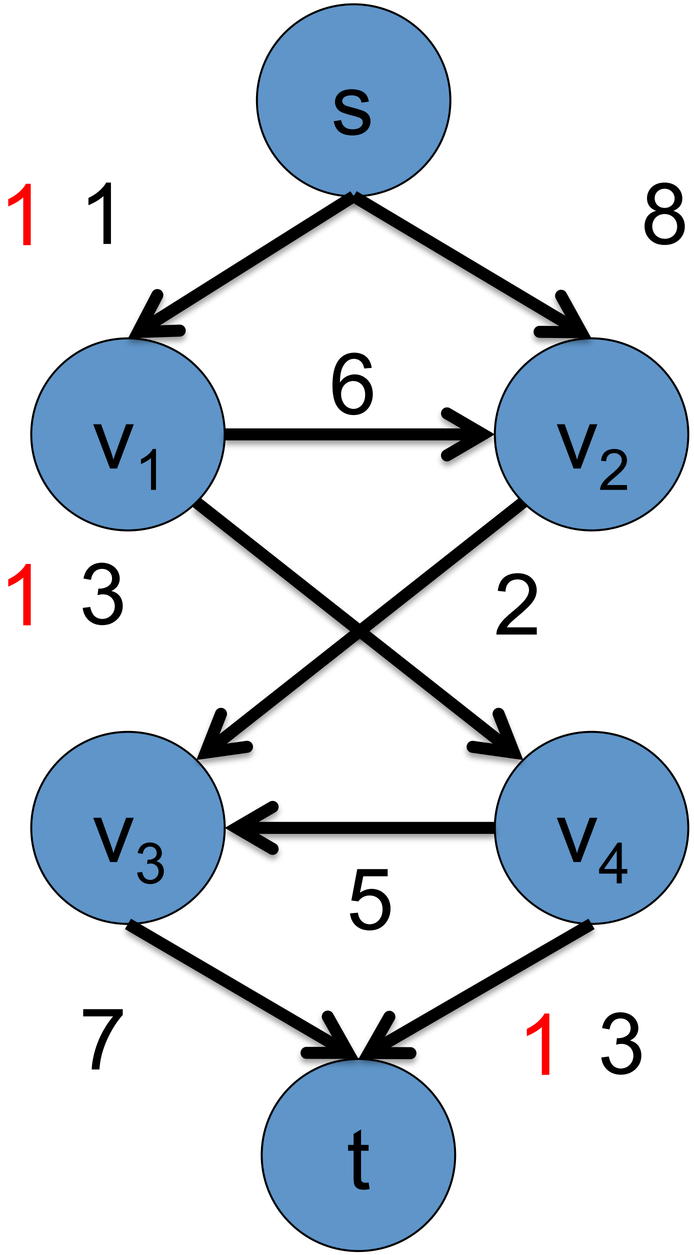


$-E_{flow}(s)$ $E_{flow}(t)$

$\Sigma_{(s,v)\in A}$ flow((s,v))

$-\ \Sigma_{(u,s)\in A}$ flow((u,s))

Value of s-t Flow



$-E_{flow}(s)$ $E_{flow}(t)$

$\Sigma_{(s,v)\in A}$ flow((s,v))

$- \Sigma_{(u,s)\in A}$ flow((u,s))

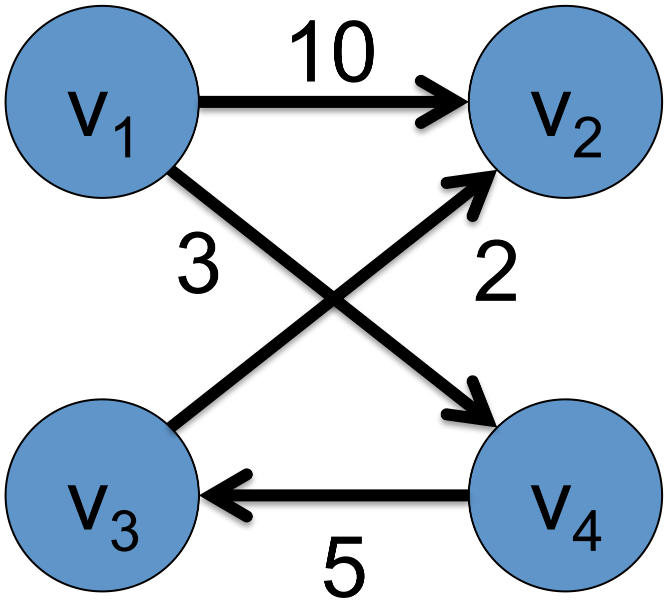
Value = 1

Outline

- Preliminaries
  - Functions and Excess Functions
  - s-t Flow
  - **s-t Cut**
  - Flows vs. Cuts
- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

Cut

D = (V, A)



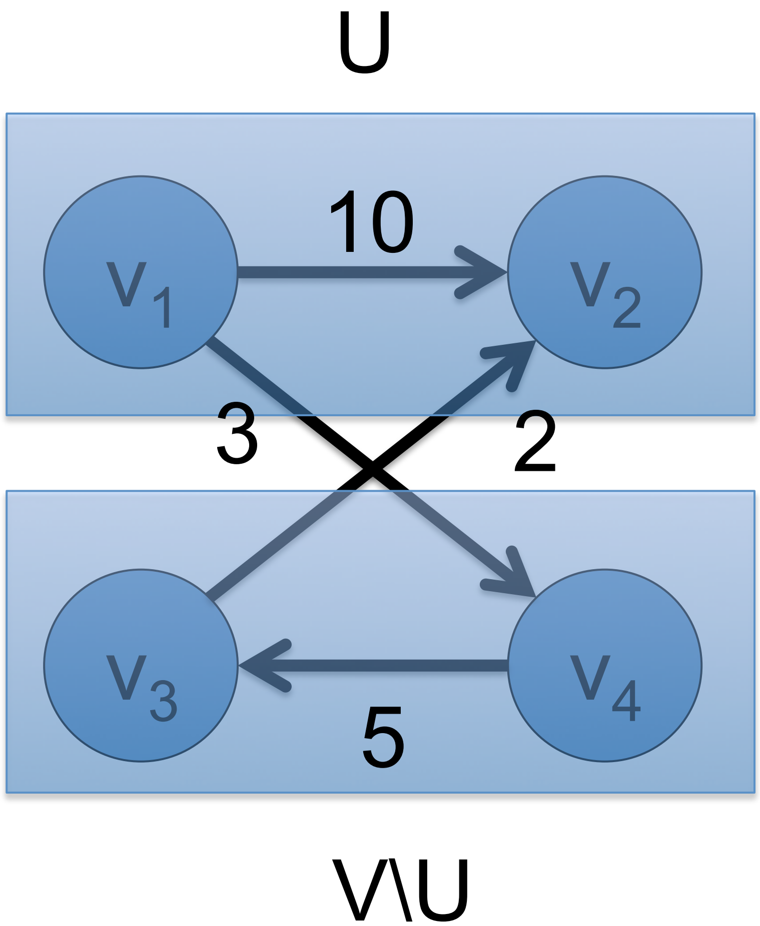
Let U be a subset of V

C is a set of arcs such that

- $(u,v) \in A$
- $u \in U$
- $v \in V \backslash U$

C is a cut in the digraph D

Cut



D = (V, A)

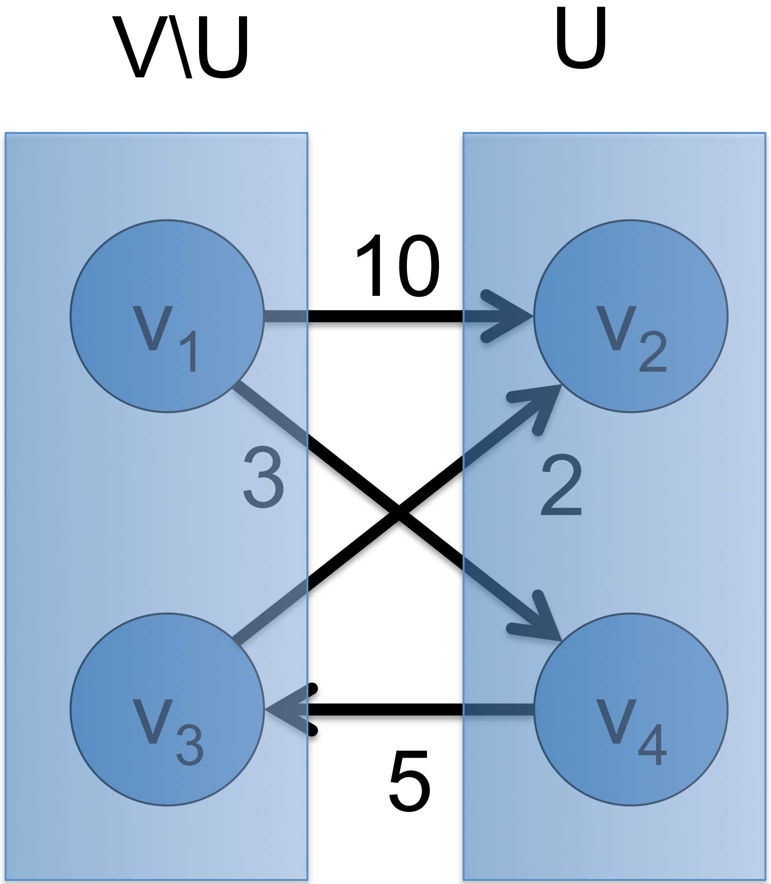
What is C?

$\{(v_1,v_2),(v_1,v_4)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

✓ $\{(v_1,v_4)\}$?

Cut



D = (V, A)

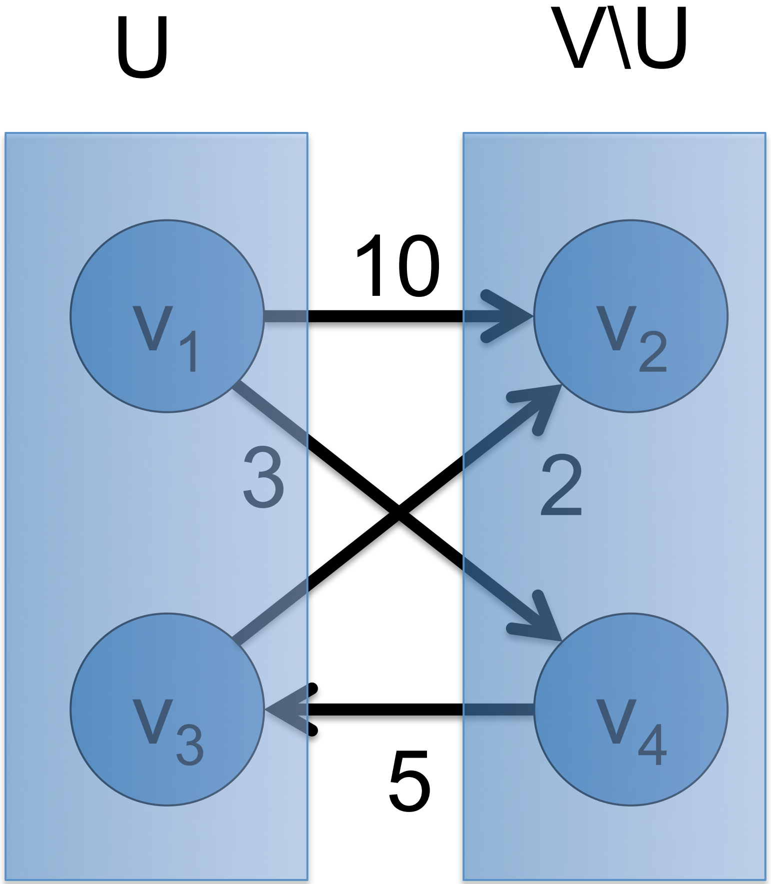
What is C?

$\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$?

✓ $\{(v_4,v_3)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

Cut



D = (V, A)

What is C?

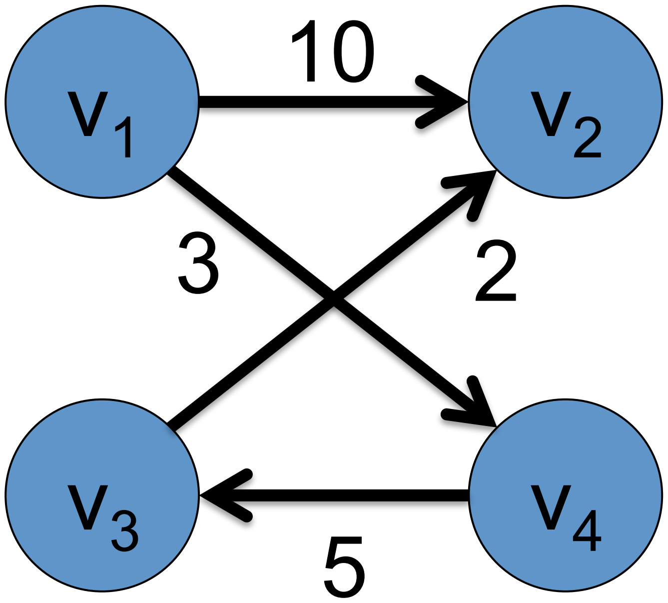
✓ $\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$?

$\{(v_3,v_2)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

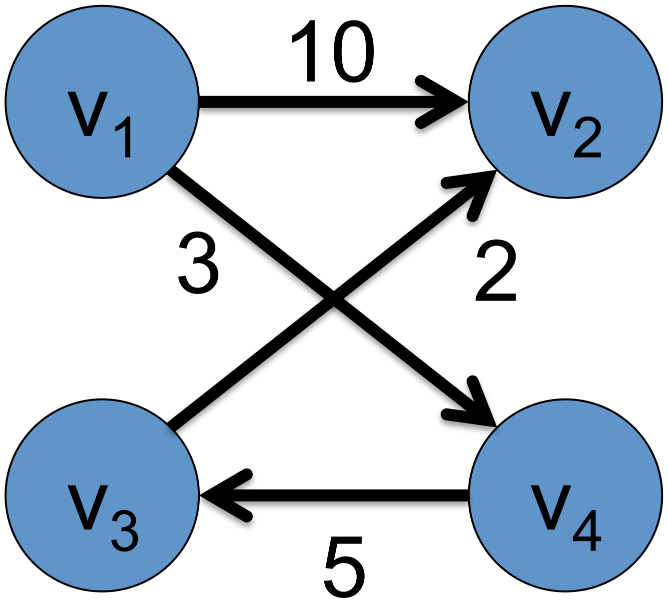
Cut

D = (V, A)



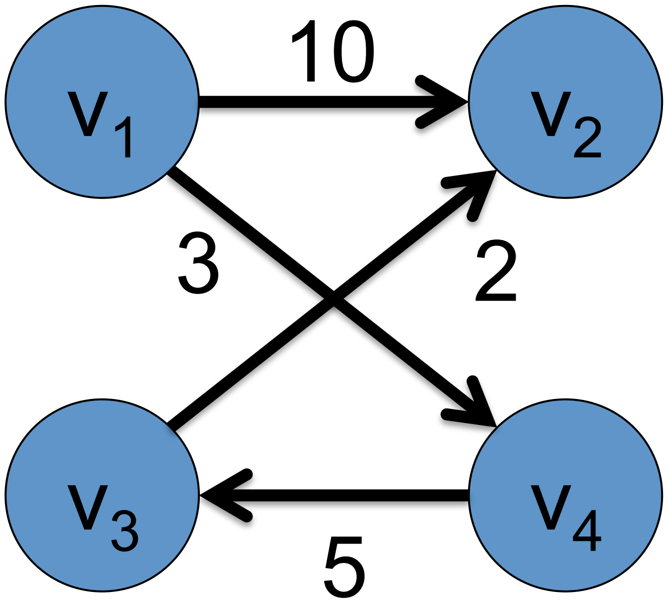
C = out-arcs(U)

Capacity of Cut



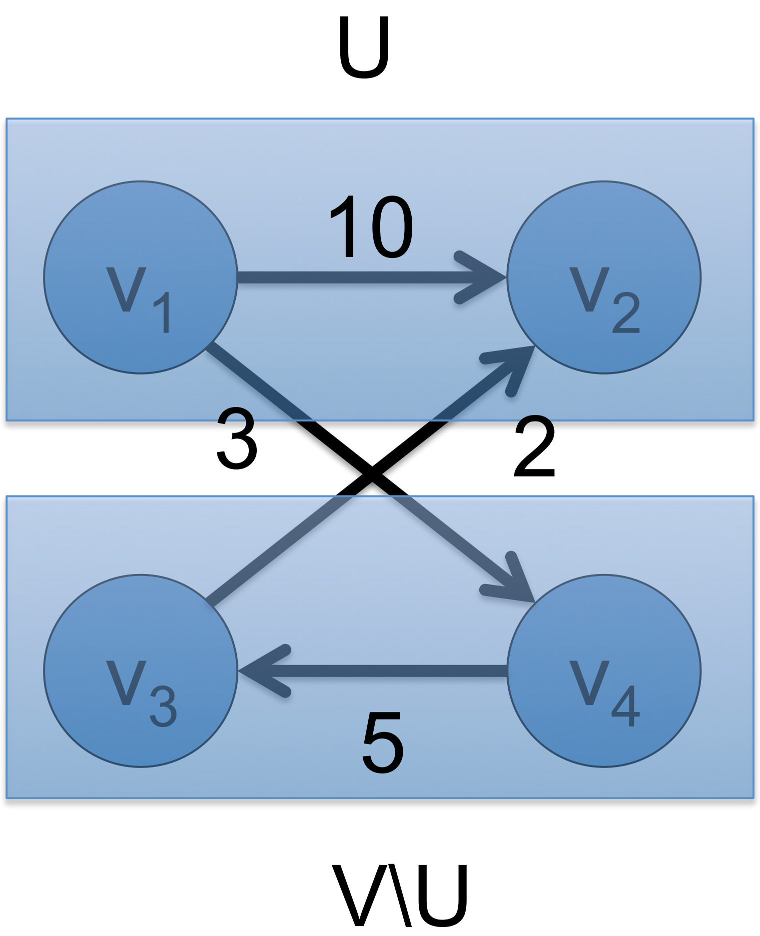
Sum of capacity of all arcs in C

Capacity of Cut



$\Sigma_{a \in C}\, c(a)$

Capacity of Cut



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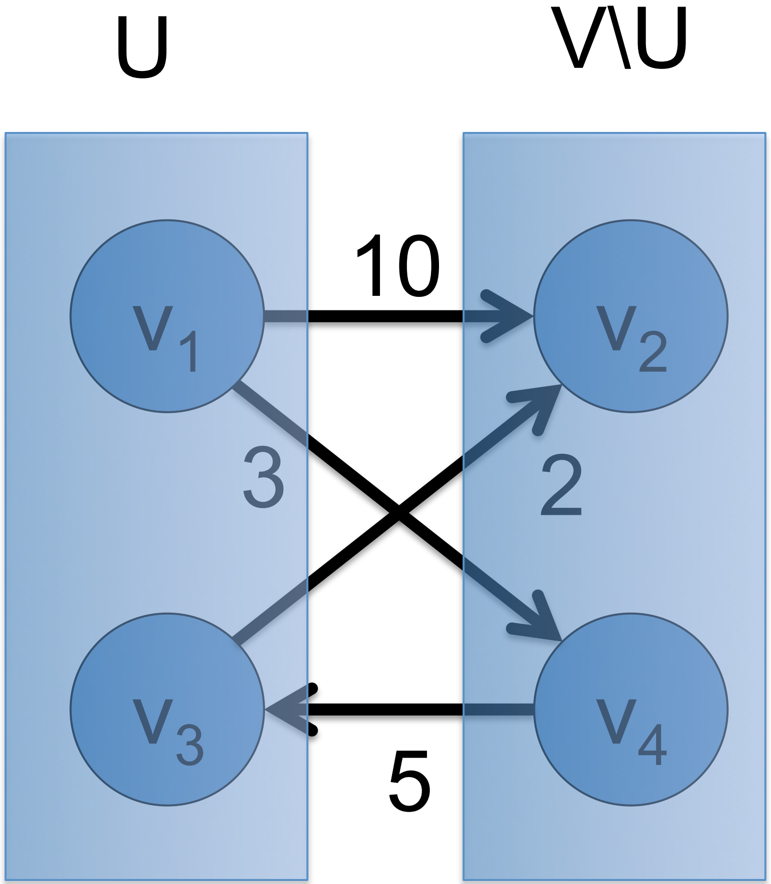
Capacity of Cut



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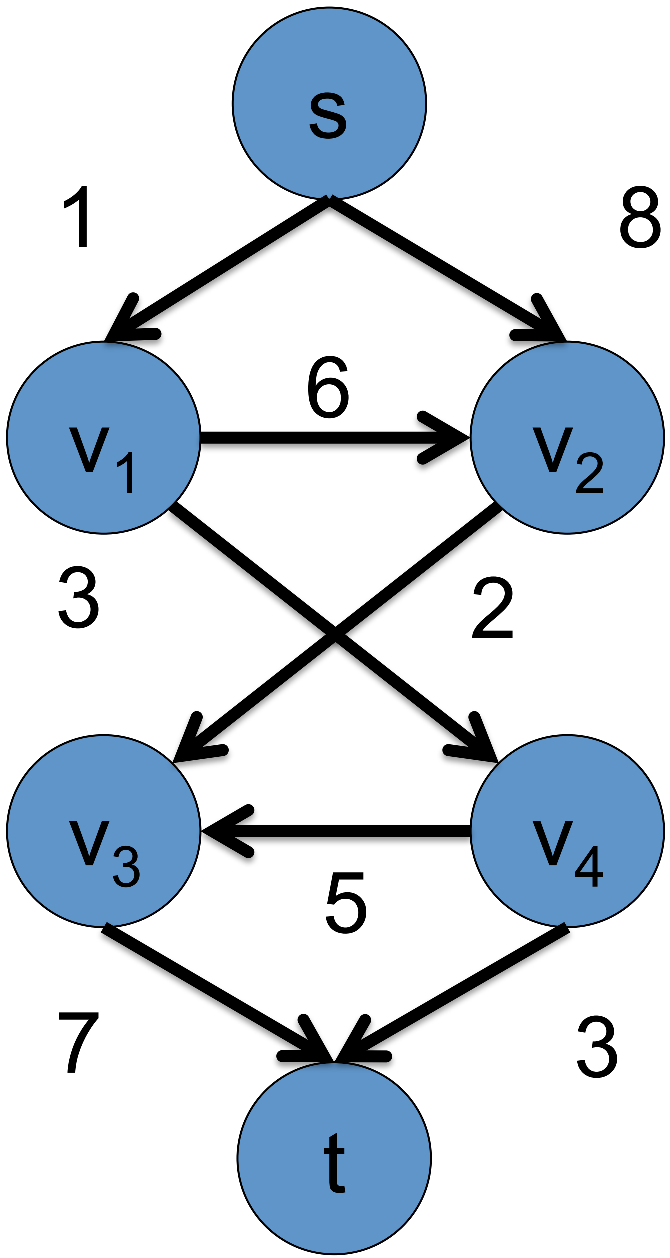
s-t Cut



D = (V, A)

A source vertex "s"

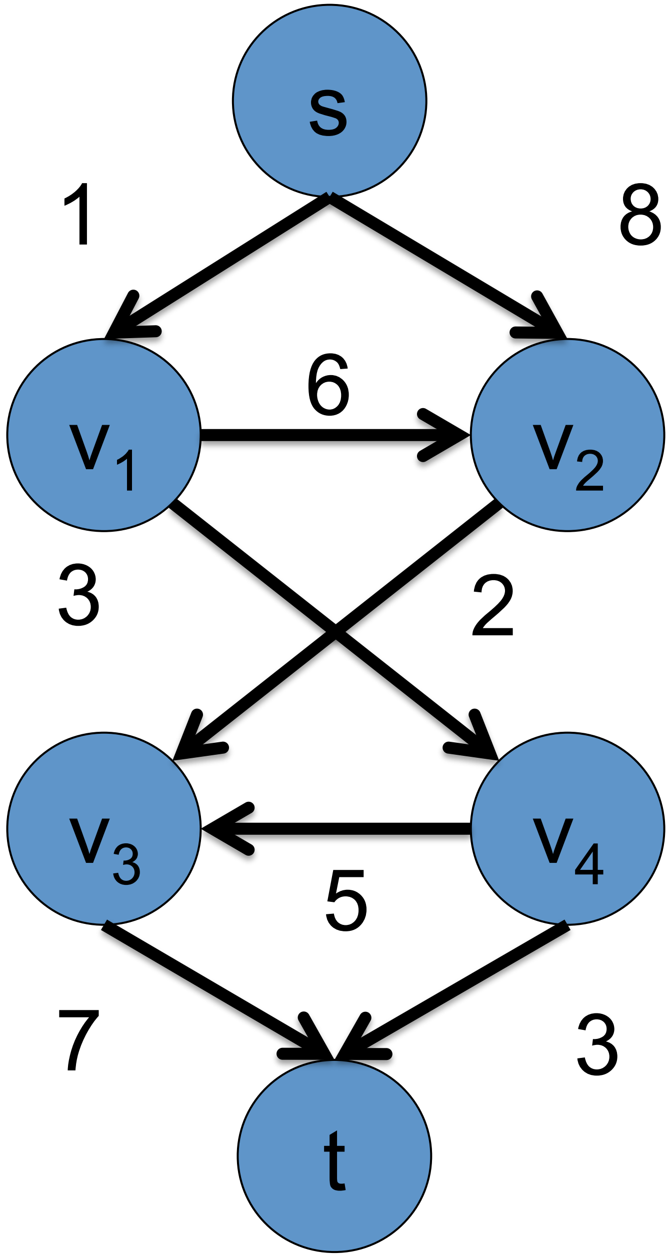
A sink vertex "t"

C is a cut such that

- $s \in U$
- $t \in V \backslash U$

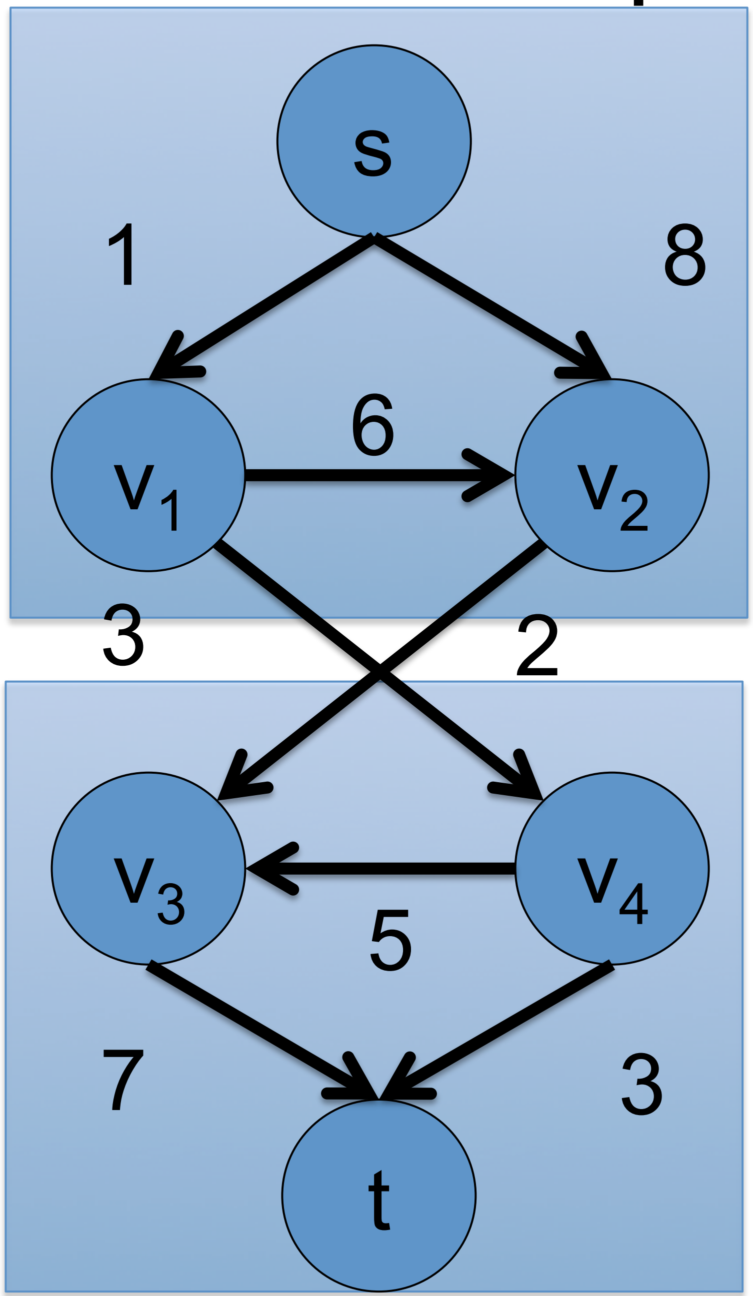
C is an s-t cut

Capacity of s-t Cut



$$\Sigma_{a \in C}\ c(a)$$

Capacity of s-t Cut



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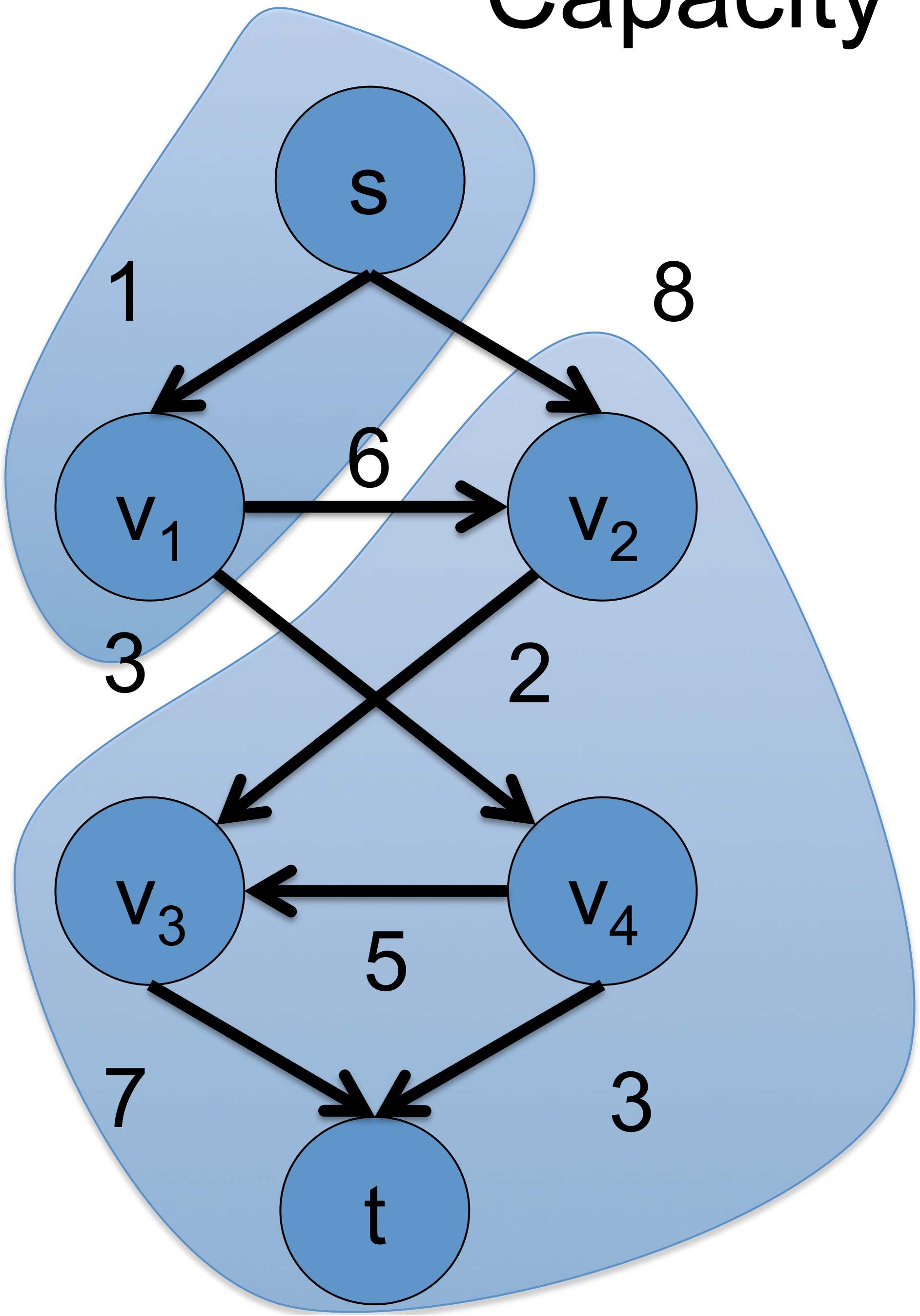
Capacity of s-t Cut



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Outline

- Preliminaries
  - s-t Flow
  - s-t Cut
  - **Flows vs. Cuts**

- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

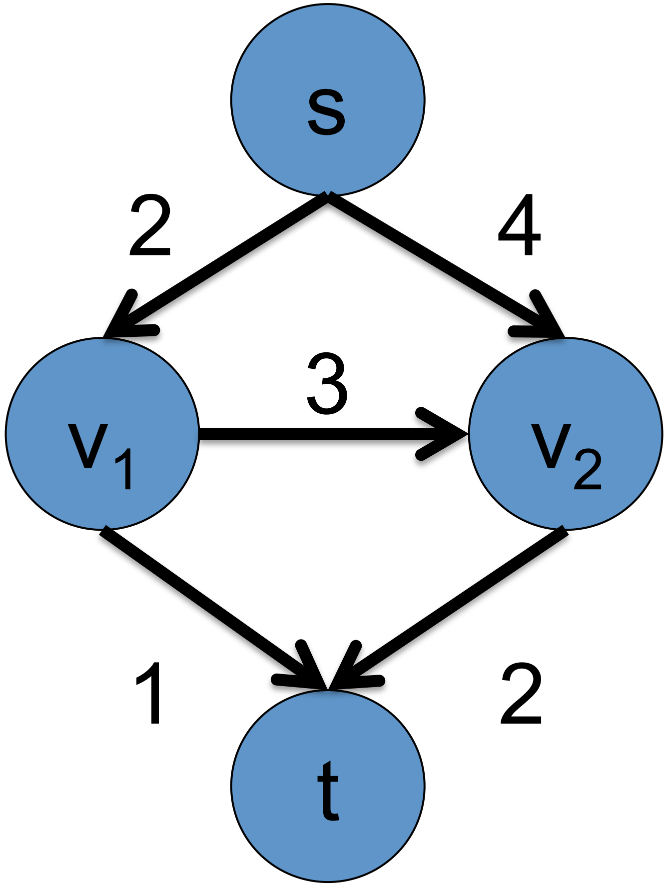
Flows vs. Cuts

(See next lecture)

Outline

- Preliminaries
- **Maximum Flow**
  - Residual Graph
  - Max-Flow Min-Cut Theorem
- Algorithms
- Energy minimization with max flow/min cut

Maximum Flow Problem



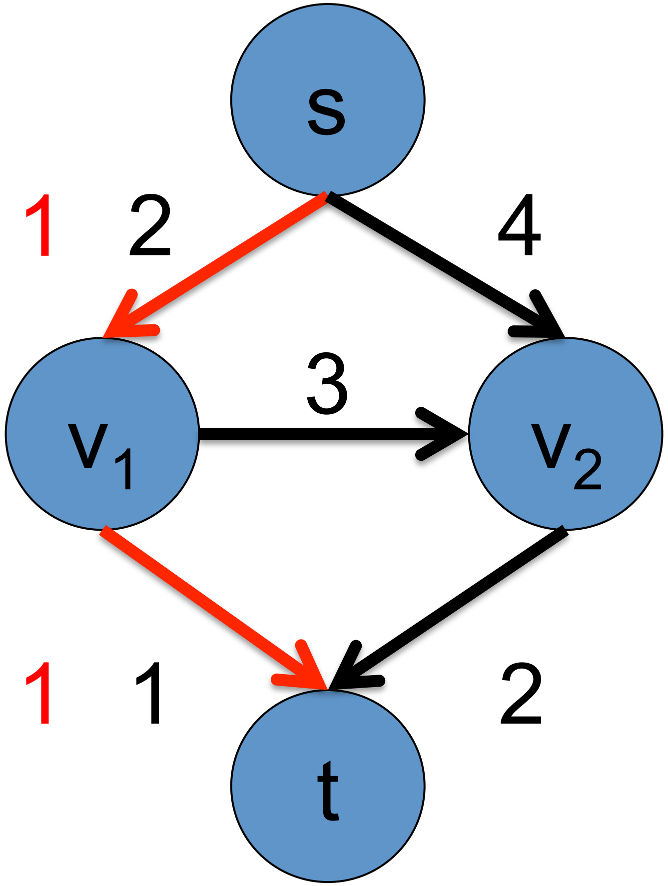
Find the flow with the maximum value !!

$$\Sigma_{(s,v)\in A} \text{flow}((s,v))$$

$$- \Sigma_{(u,s)\in A} \text{flow}((u,s))$$

First suggestion to solve this problem !!

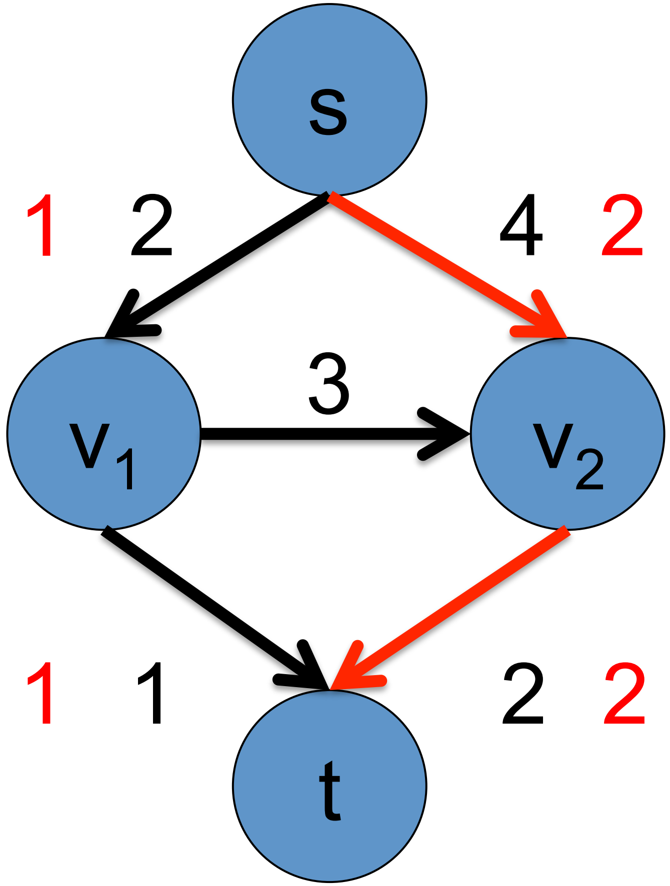
Passing Flow through s-t Paths



Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

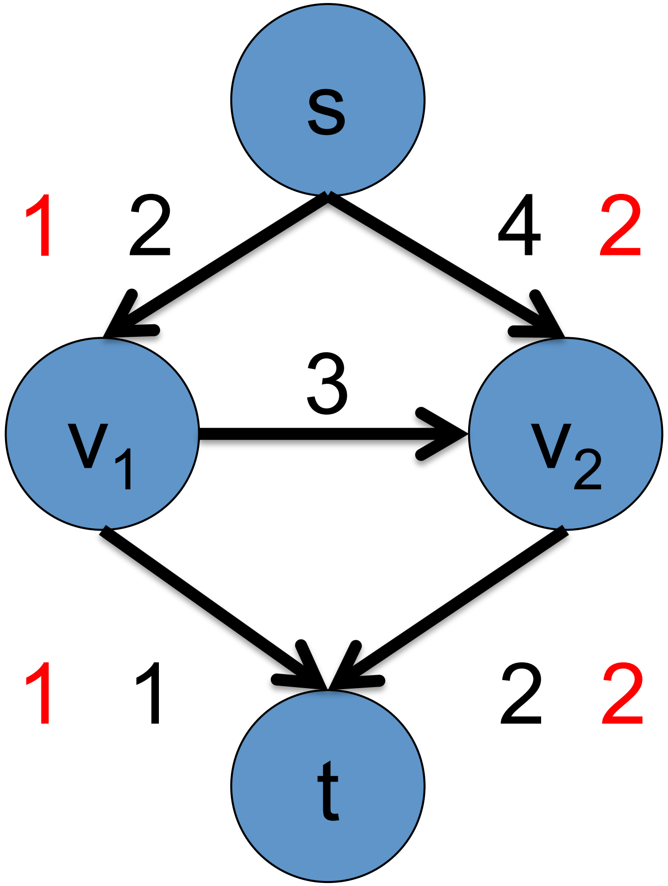
Passing Flow through s-t Paths



Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

Passing Flow through s-t Paths

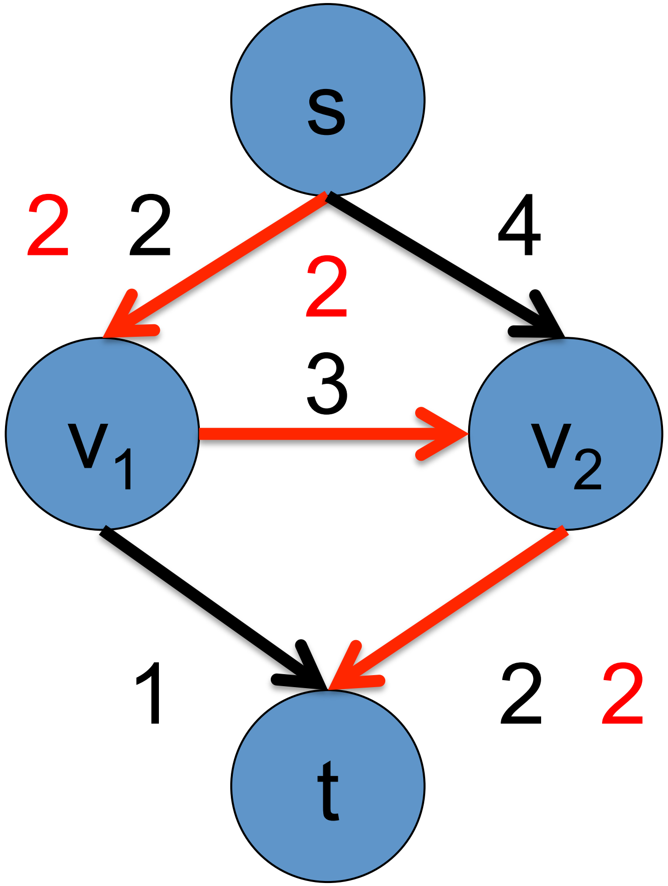


Find an s-t path where flow(a) < c(a) for all arcs

No more paths. Stop.

Will this give us maximum flow? **NO !!!**

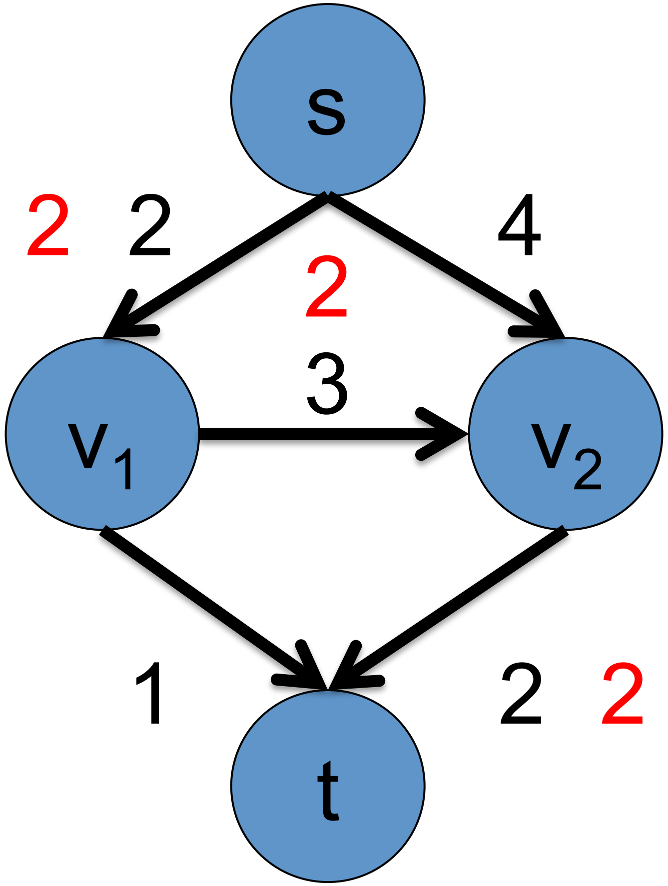
Passing Flow through s-t Paths



Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

Passing Flow through s-t Paths



Find an s-t path where flow(a) < c(a) for all arcs

No more paths. Stop.

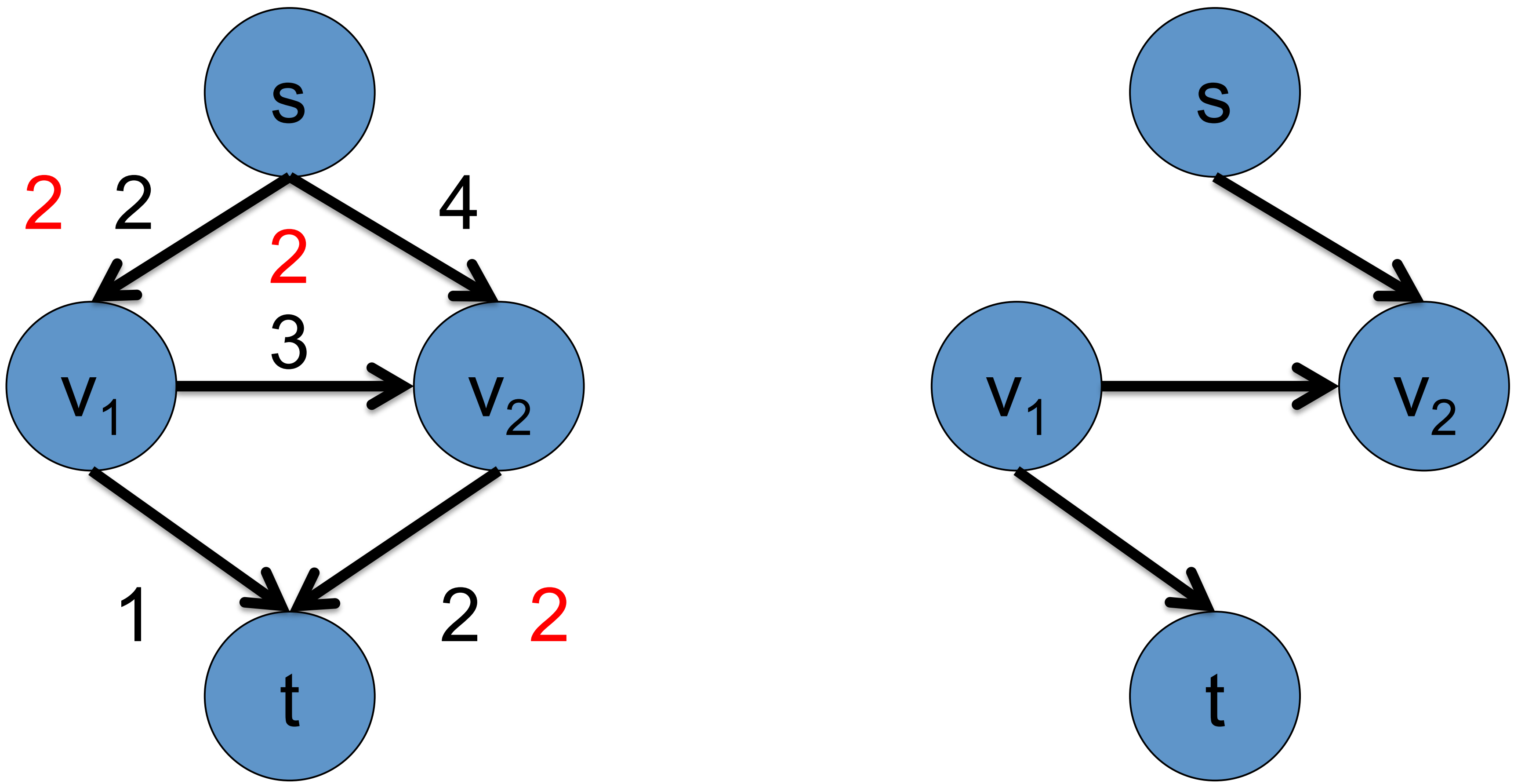
Another method?

Incorrect Answer !!

Outline

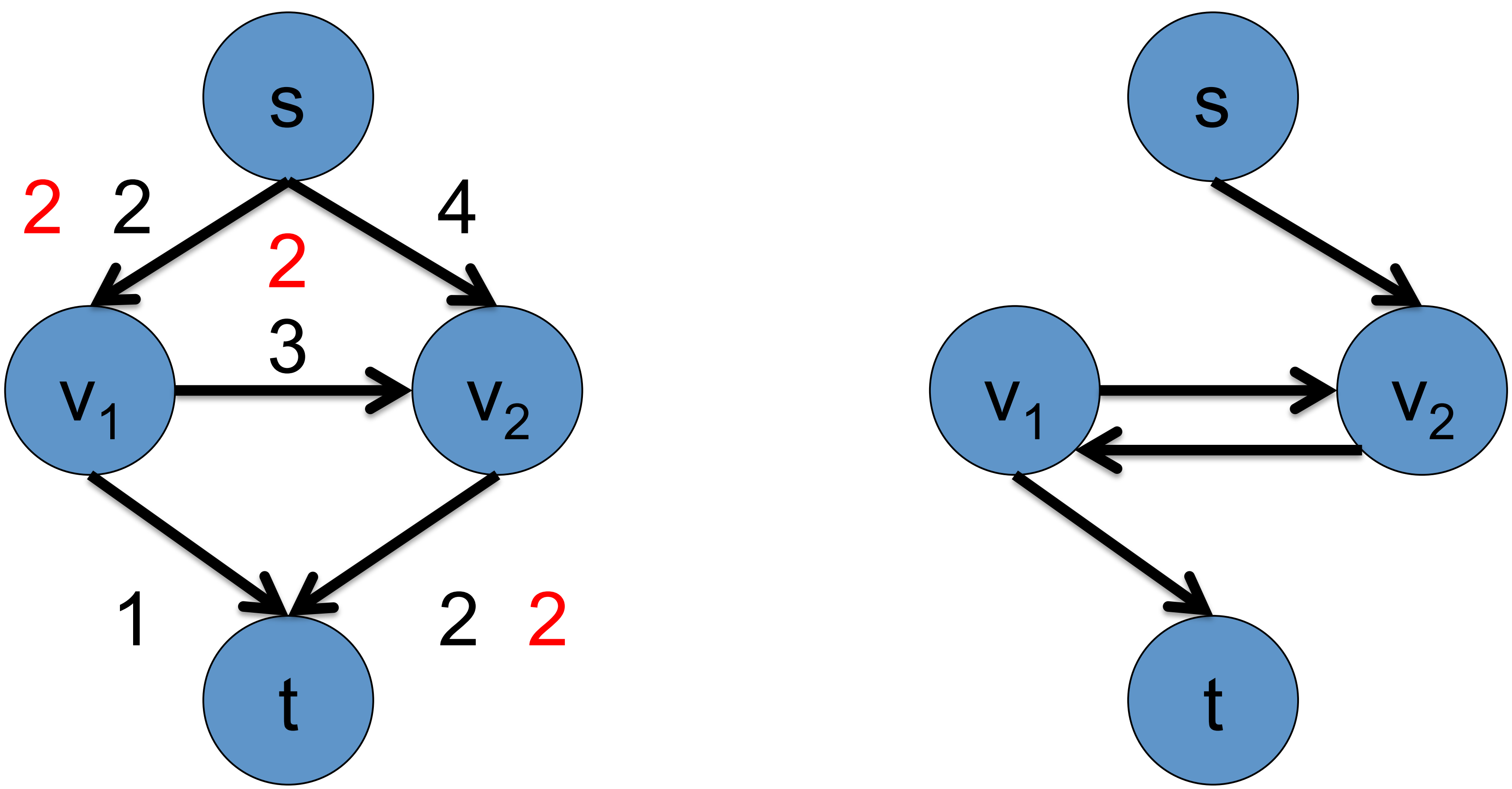
- Preliminaries
- Maximum Flow
  - **Residual Graph**
  - Max-Flow Min-Cut Theorem
- Algorithms
- Energy minimization with max flow/min cut

Residual Graph



Arcs where flow(a) < c(a)

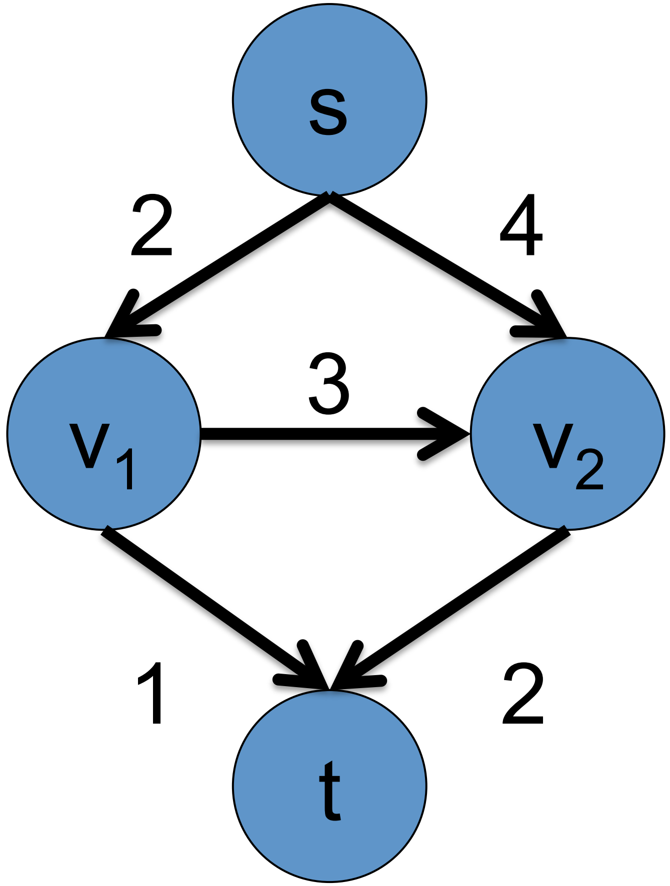
Residual Graph

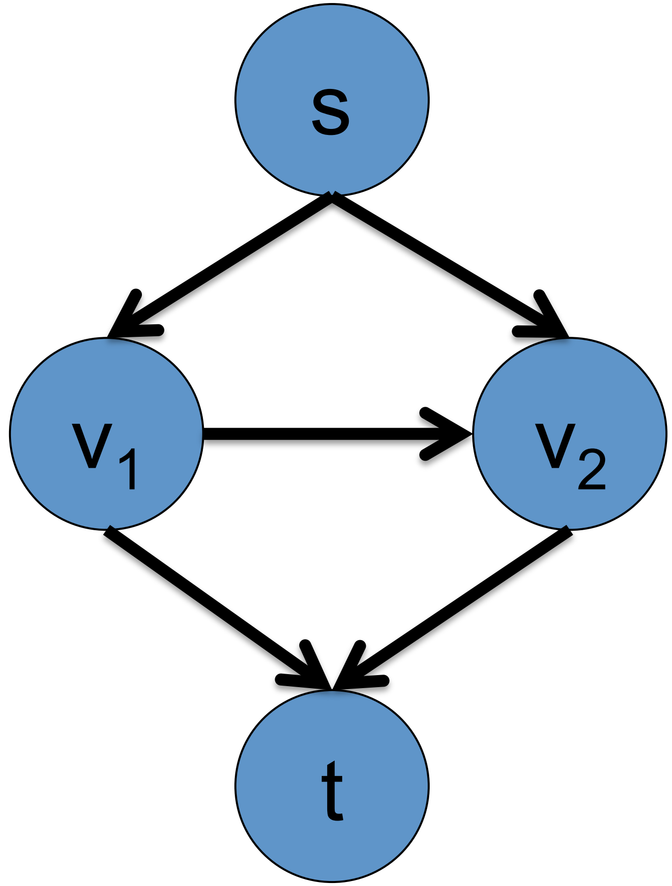


Including arcs to s and from t is not necessary

Inverse of arcs where flow(a) > 0

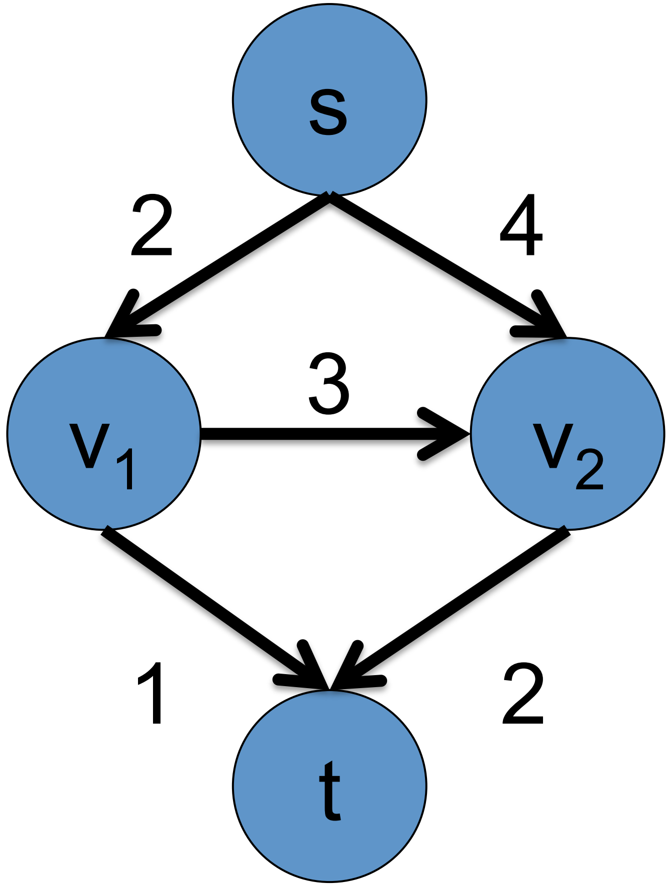
Maximum Flow using Residual Graphs

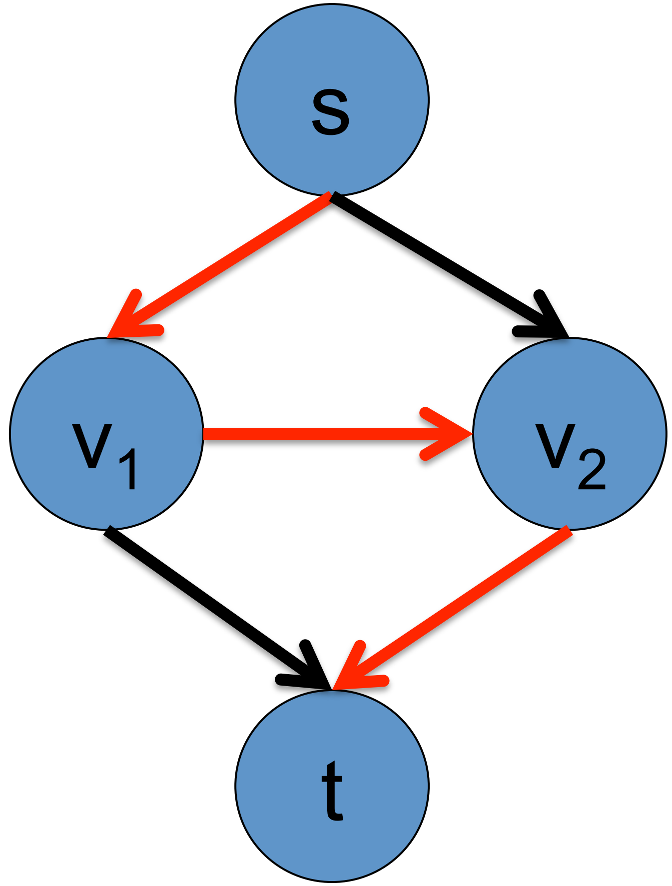




Start with zero flow.

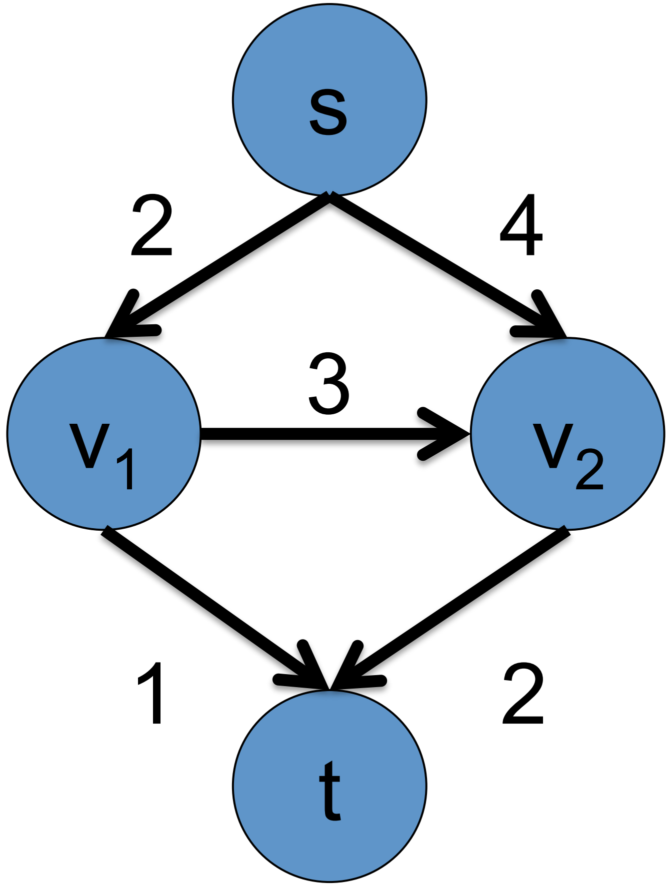
Maximum Flow using Residual Graphs

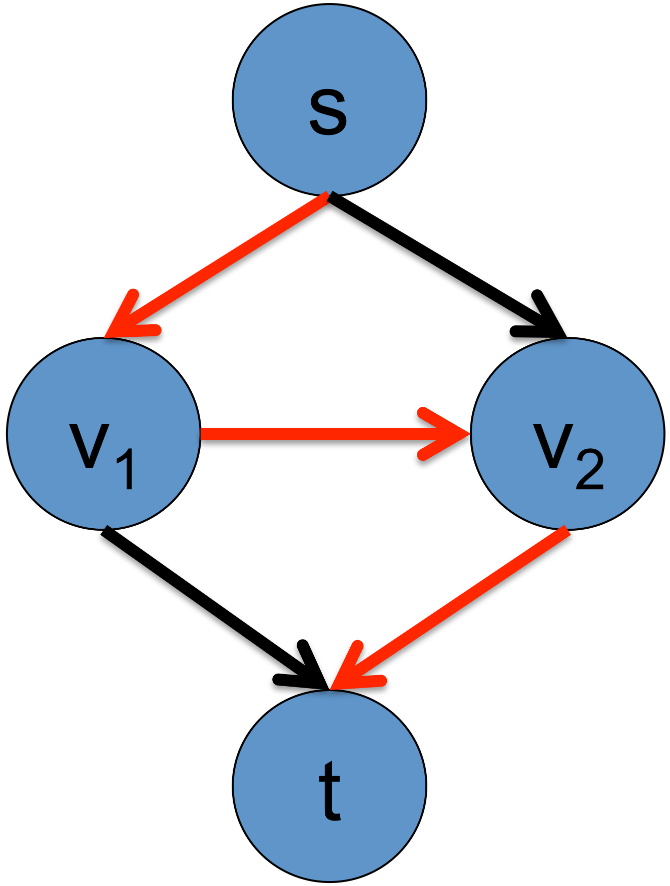




Find an s-t path in the residual graph.

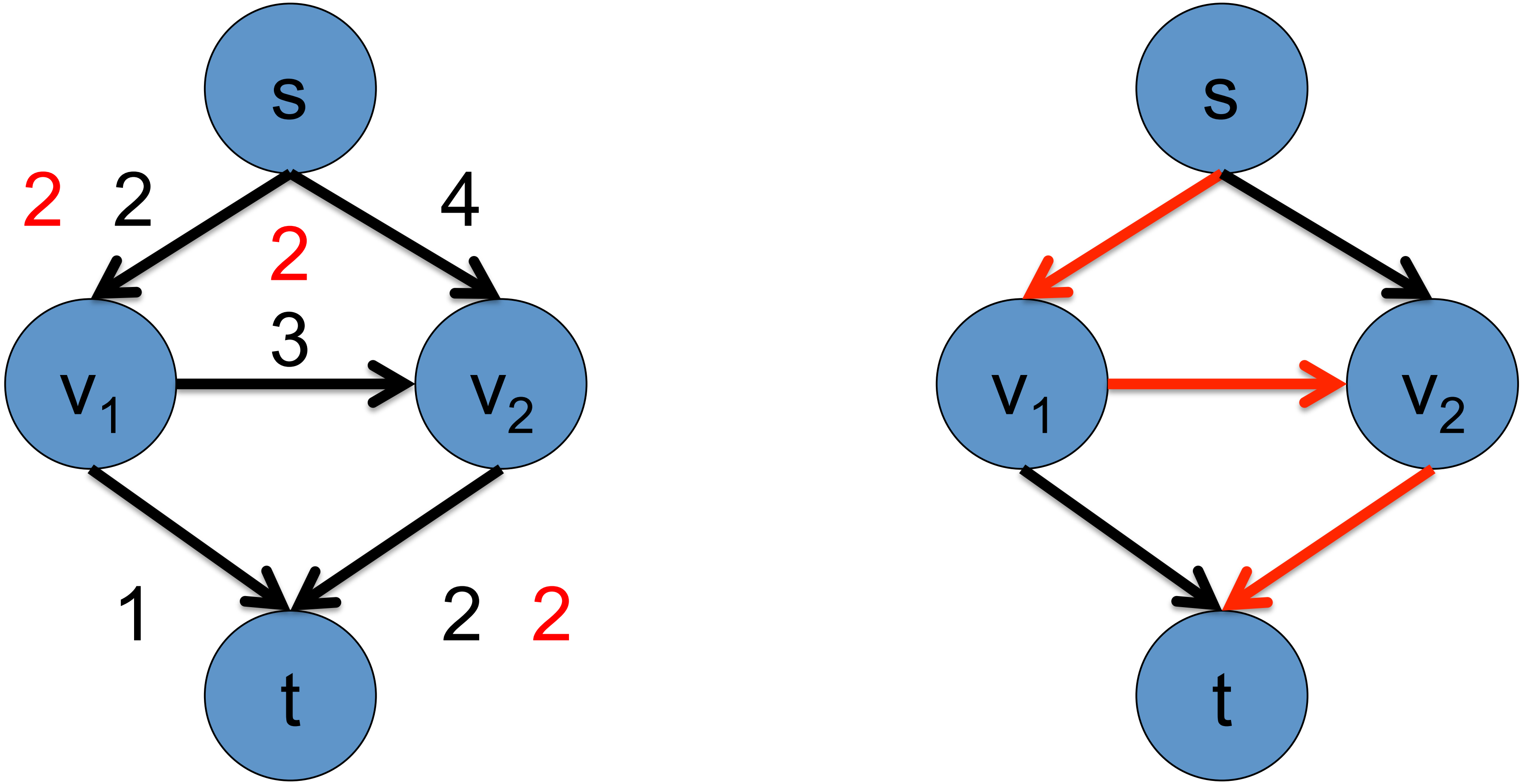
Maximum Flow using Residual Graphs





For inverse arcs in path, subtract flow K.

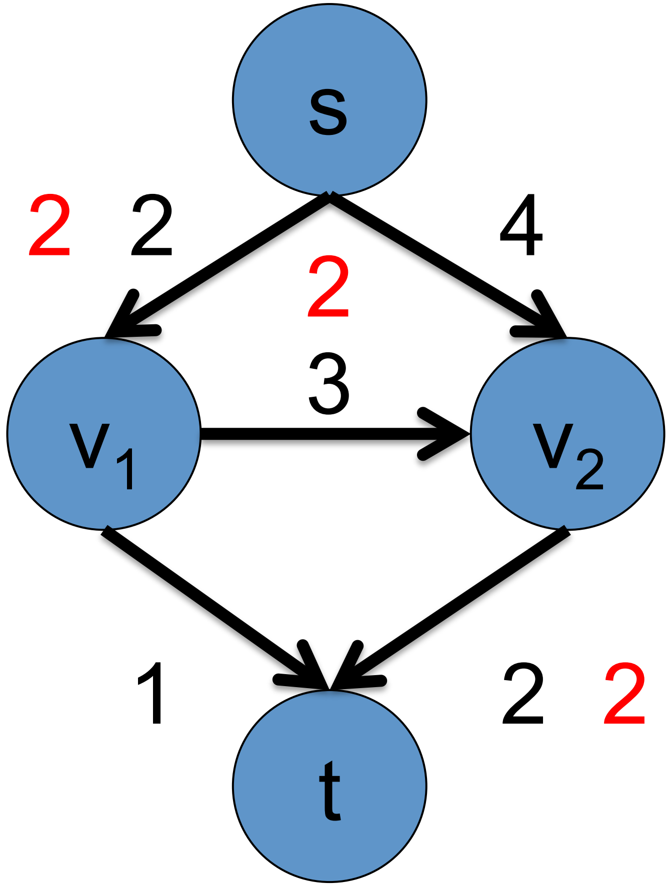
Maximum Flow using Residual Graphs

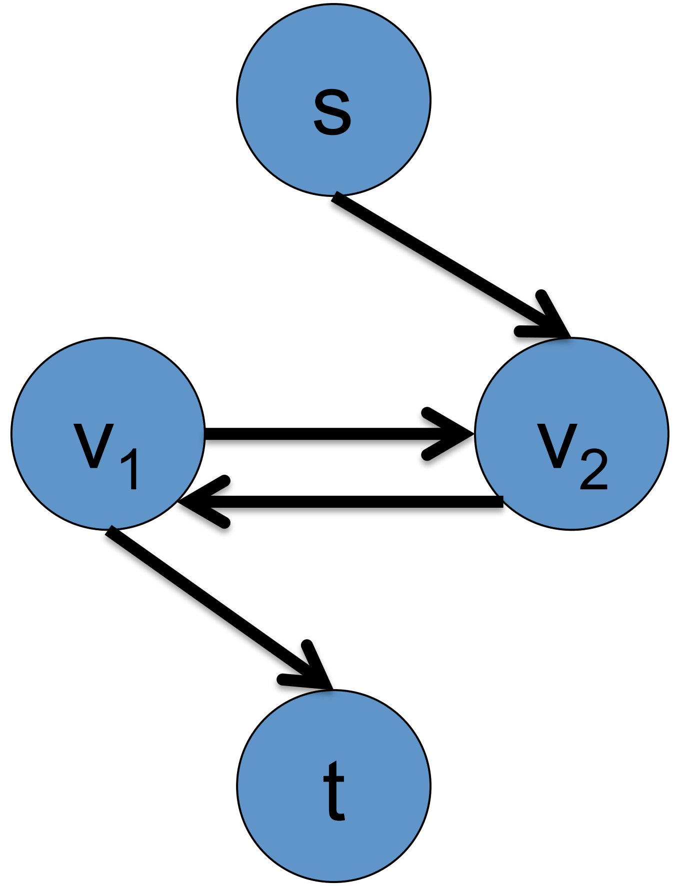


Choose maximum allowable value of K.

For forward arcs in path, add flow K.

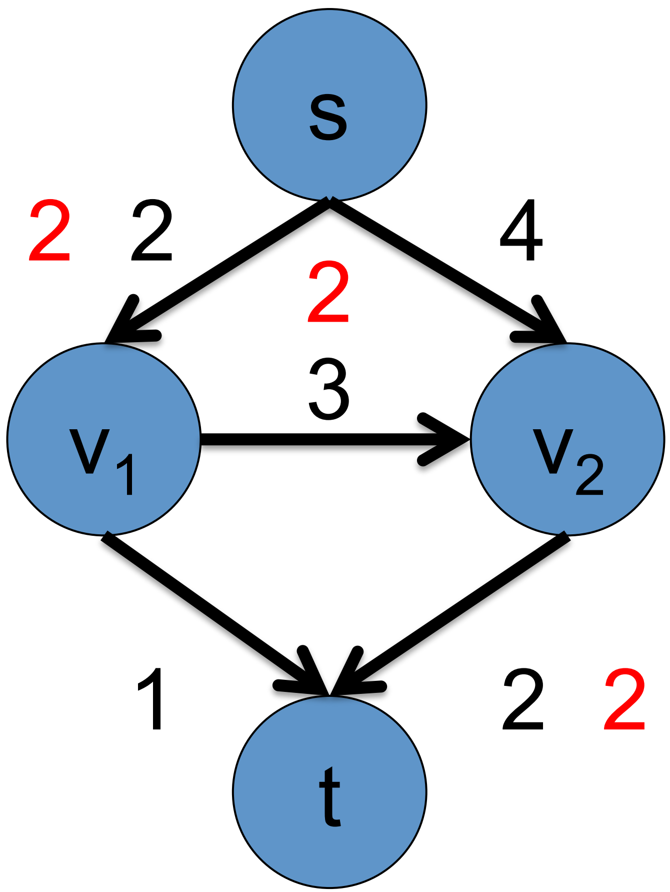
Maximum Flow using Residual Graphs

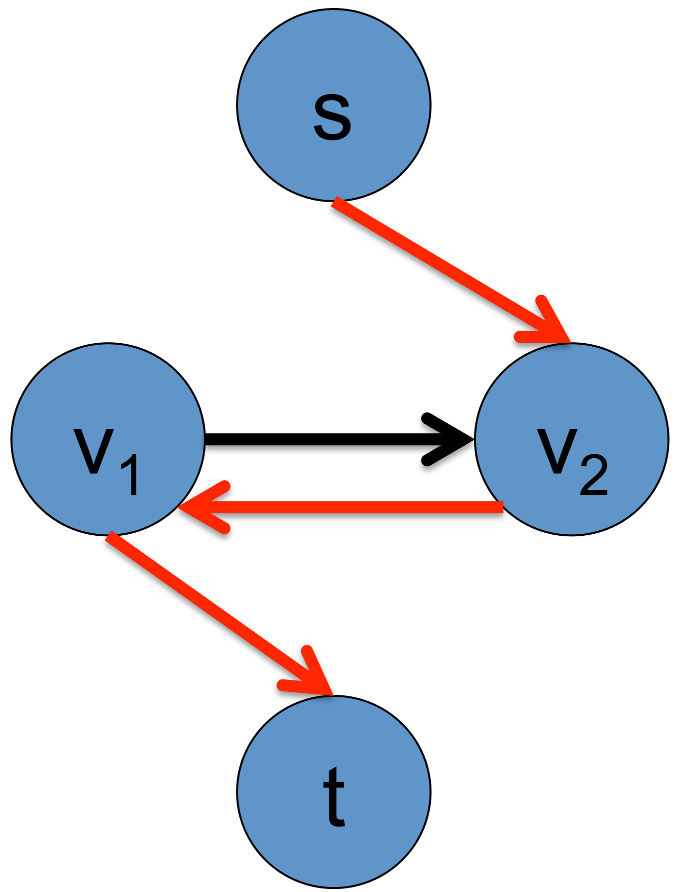




Update the residual graph.

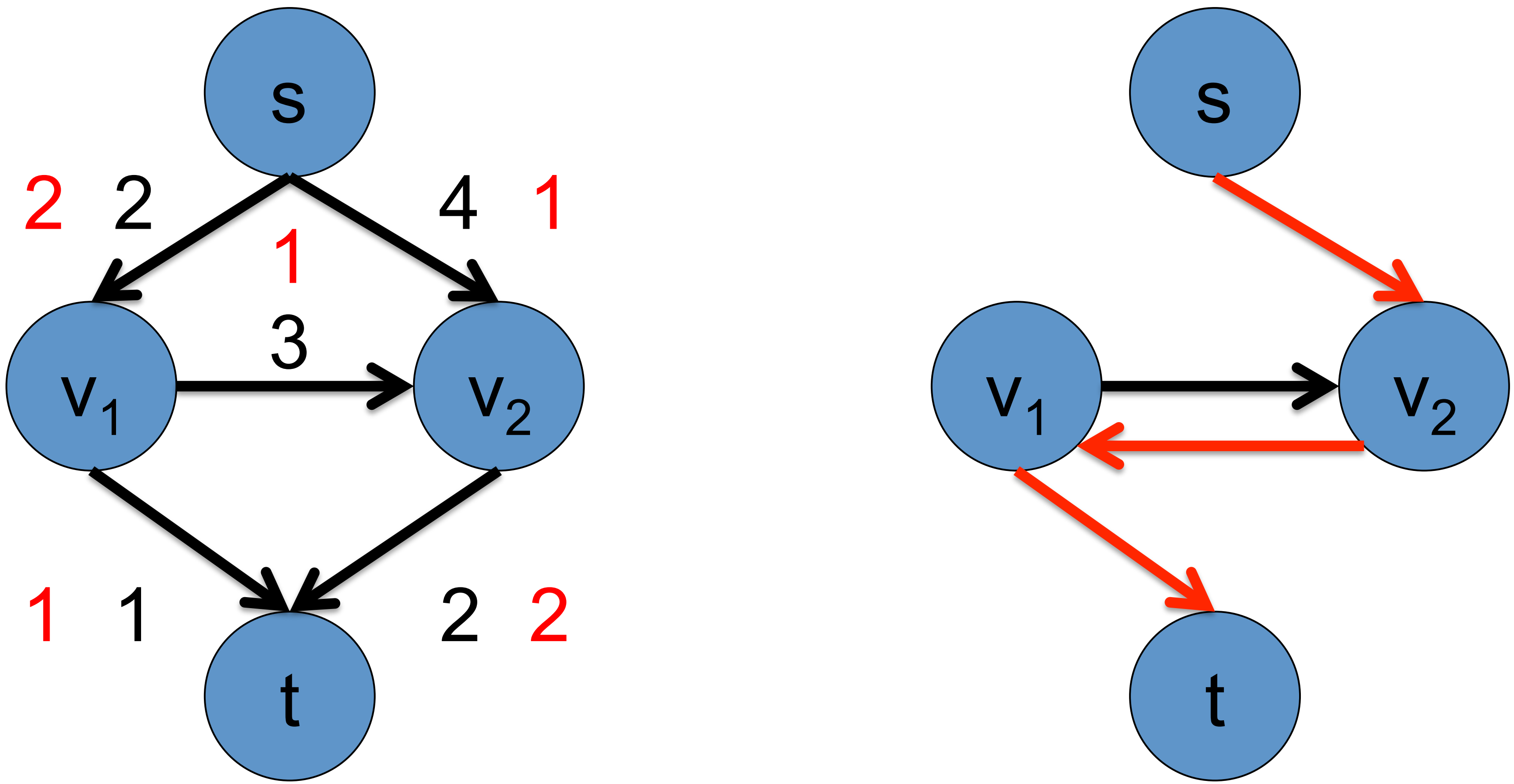
Maximum Flow using Residual Graphs





Find an s-t path in the residual graph.

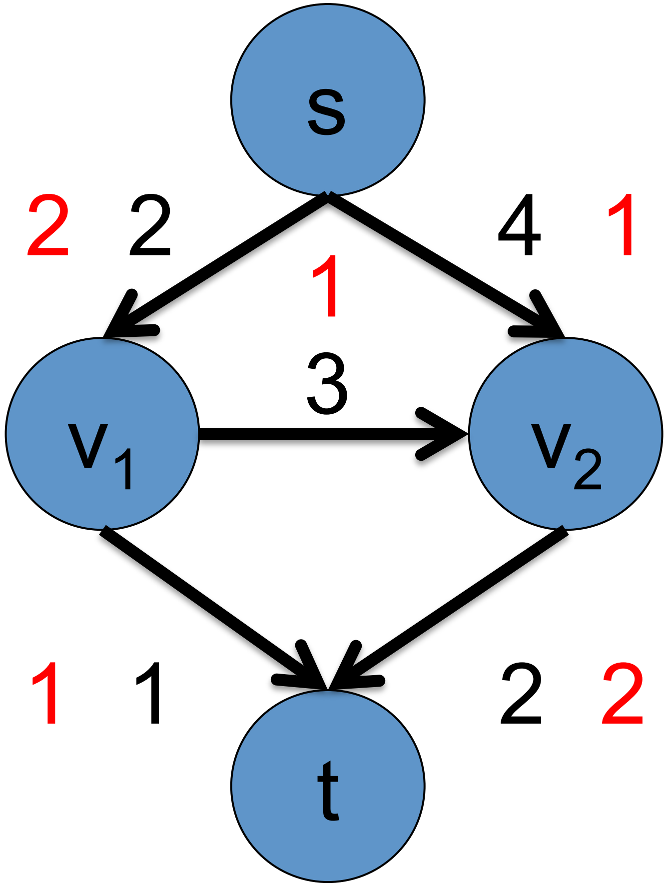
Maximum Flow using Residual Graphs

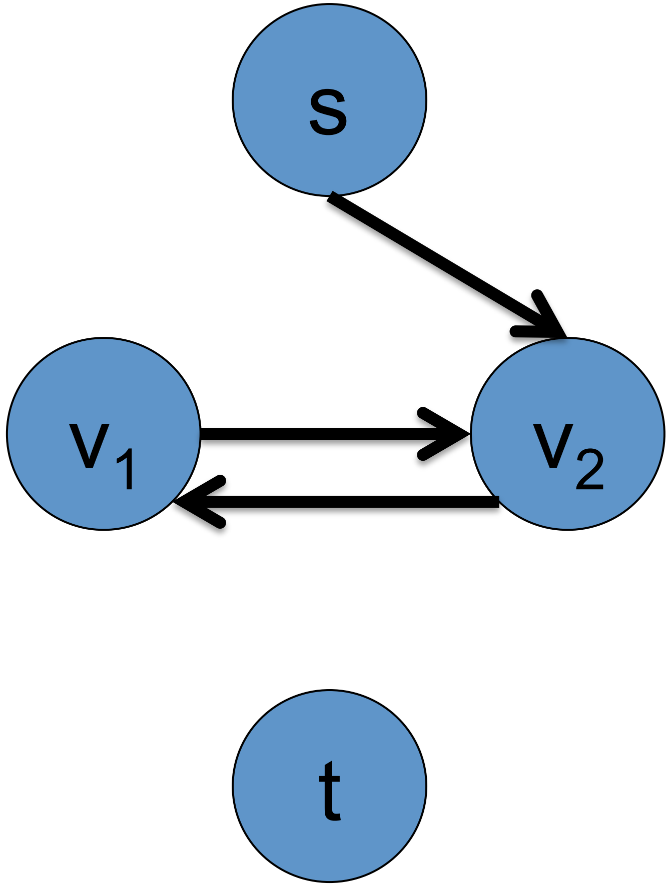


Choose maximum allowable value of K.

Add K to (s,v_2) and (v_1,t). Subtract K from (v_1,v_2).

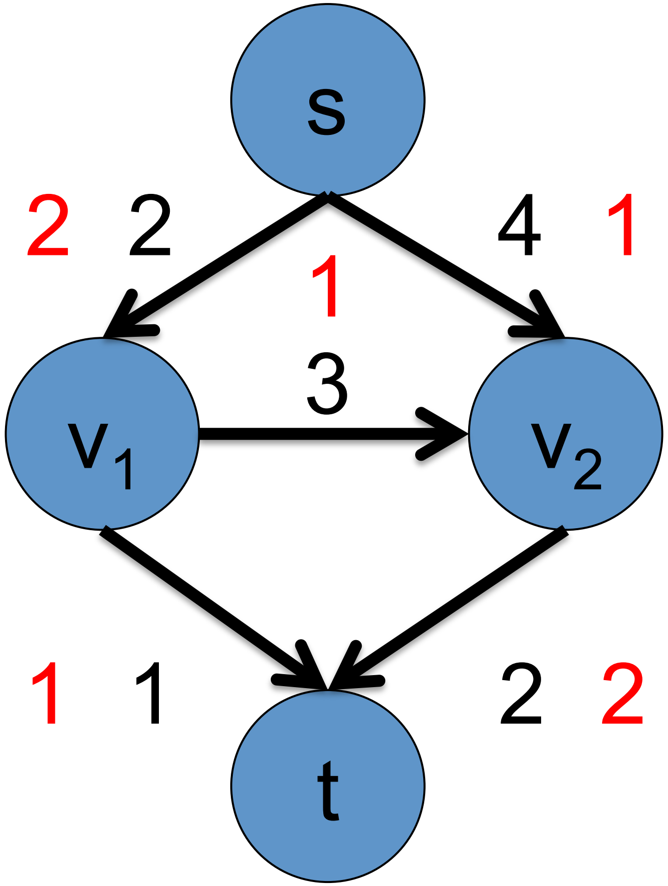
Maximum Flow using Residual Graphs

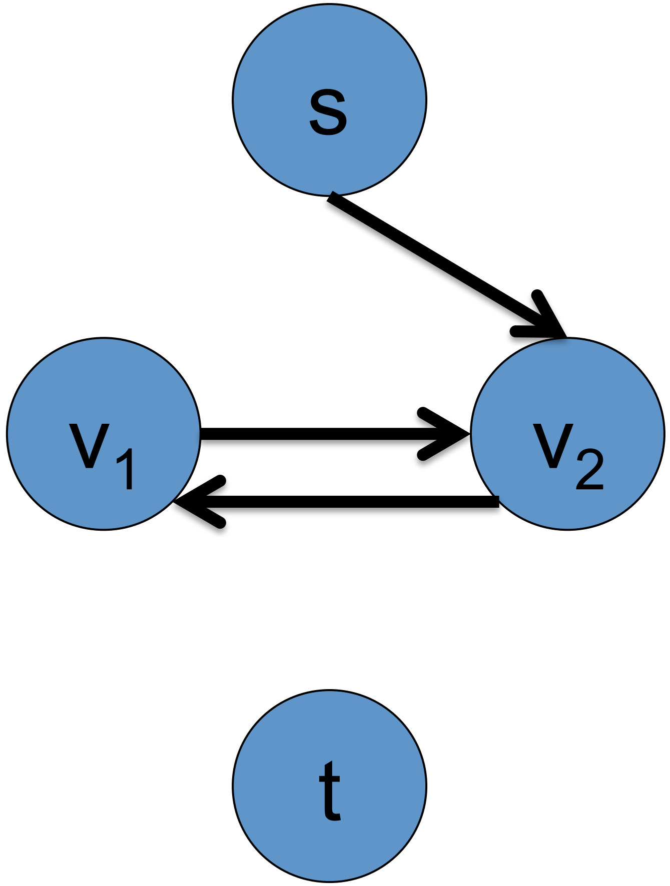




Update the residual graph.

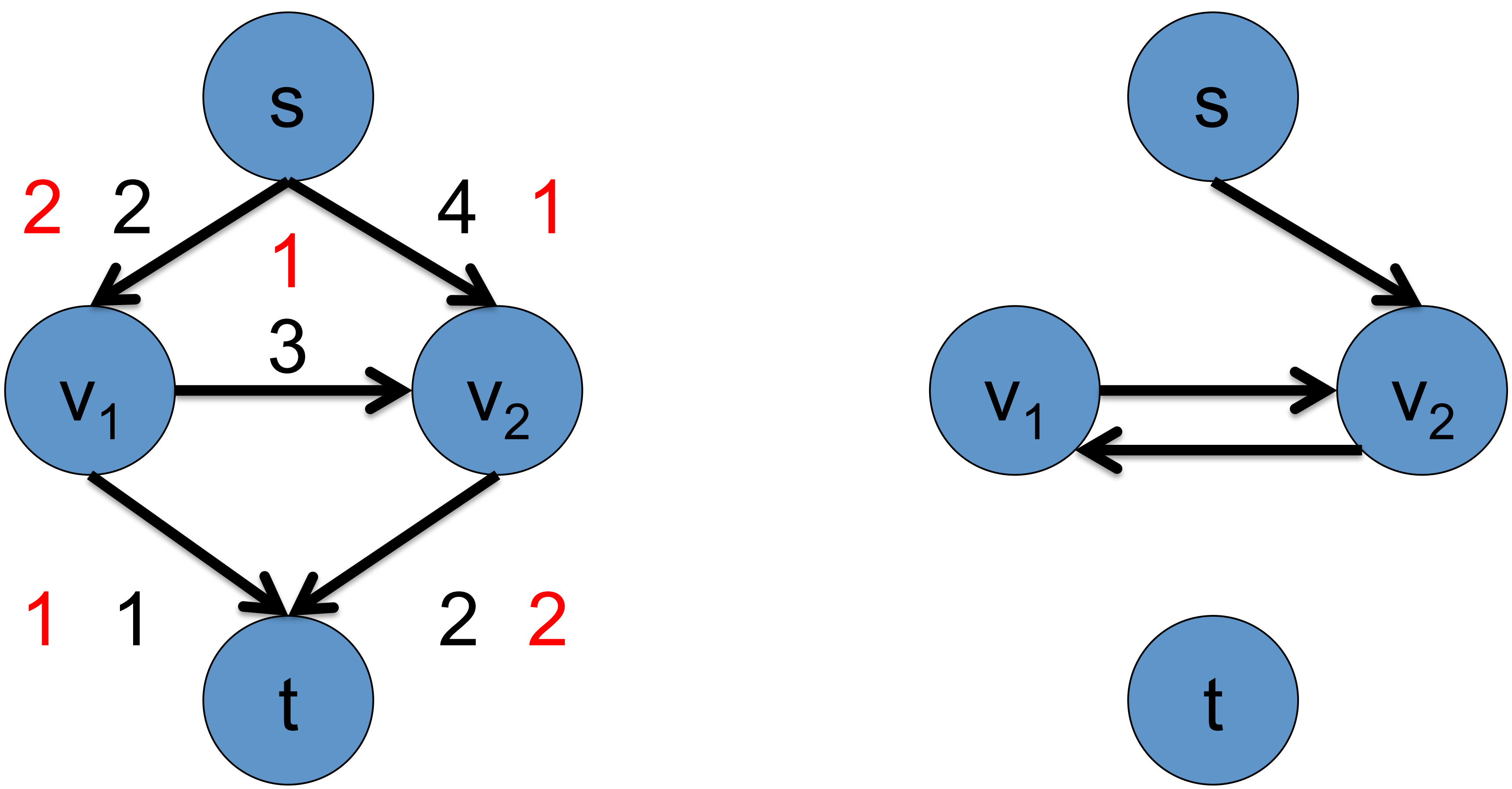
Maximum Flow using Residual Graphs





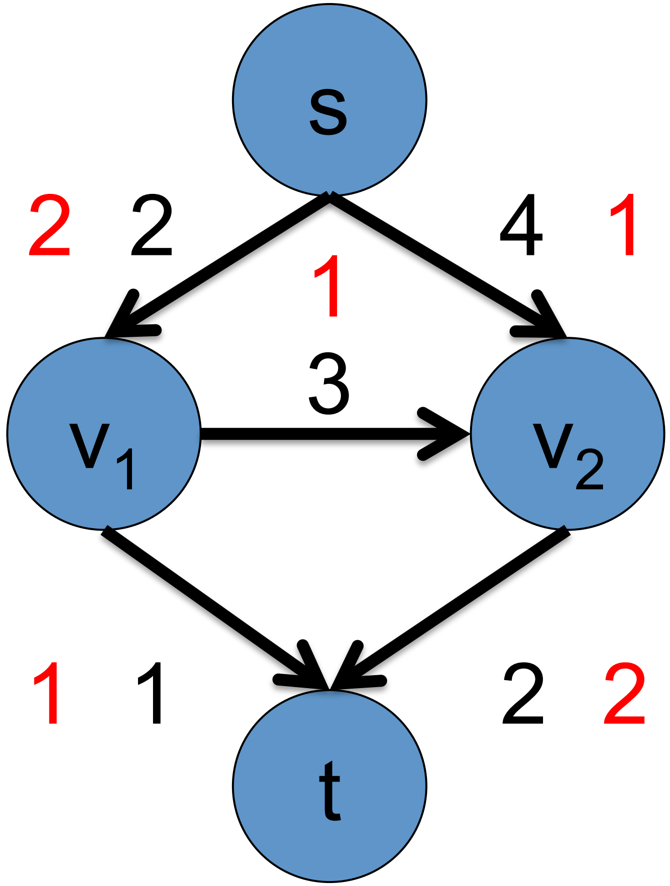
Find an s-t path in the residual graph.

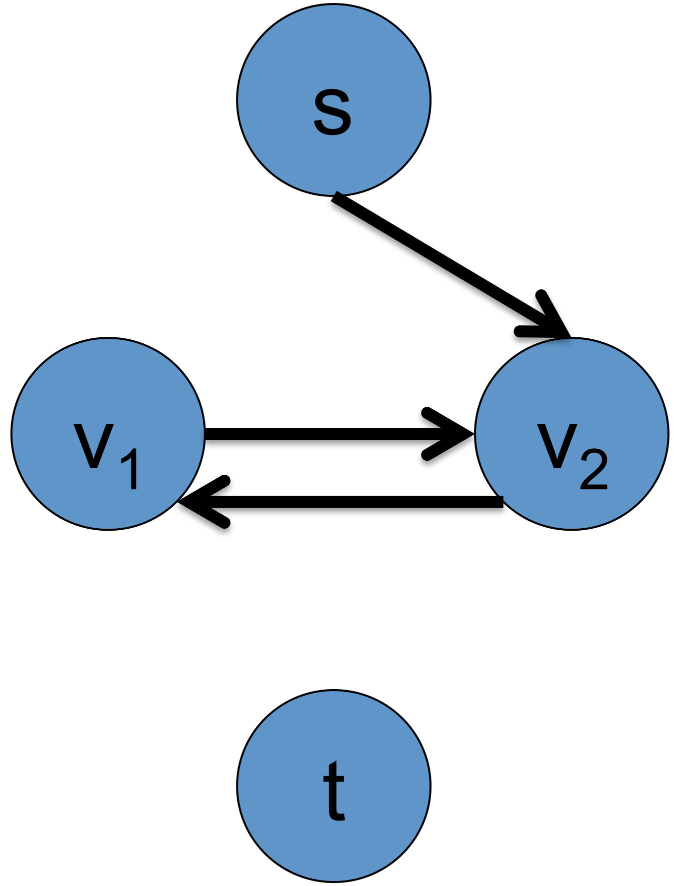
Maximum Flow using Residual Graphs



No more s-t paths. Stop.

Maximum Flow using Residual Graphs





Correct Answer.