# Graphical Models
# Discrete Inference and Learning
# Lecture 3 (contd.)

MVA

2020 – 2021

http://thoth.inrialpes.fr/~alahari/disinflearn

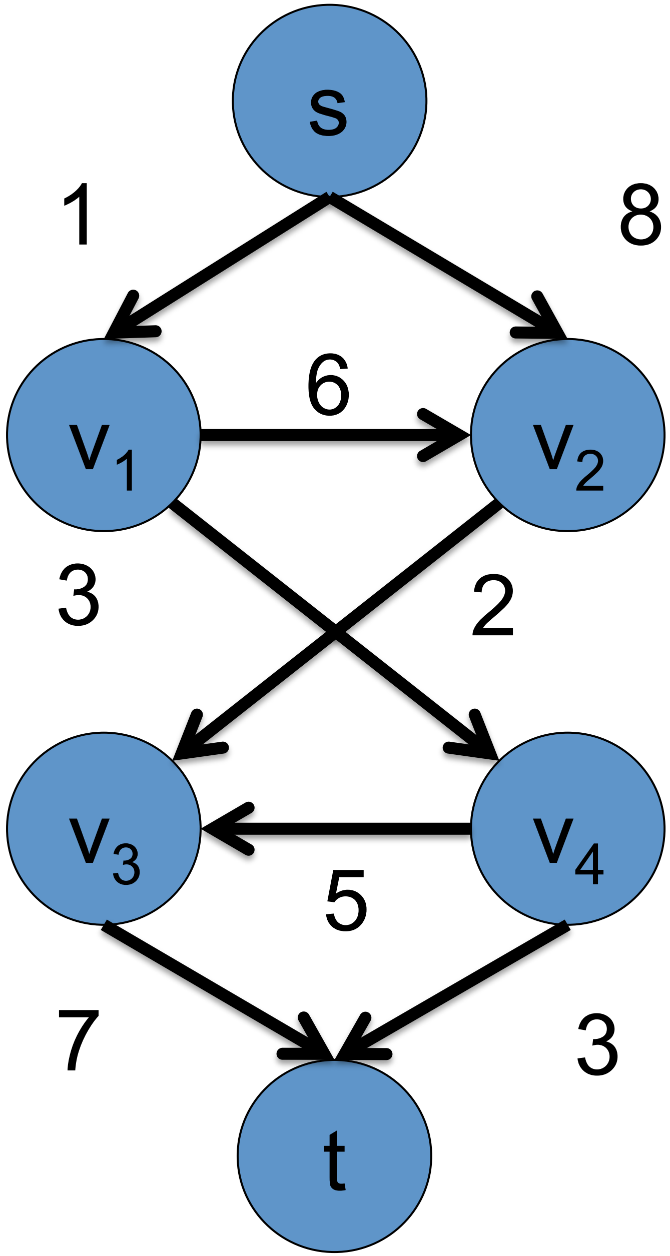
Slides based on material from M. Pawan Kumar

# Outline

- Preliminaries
  - **s-t Flow**
  - s-t Cut
  - Flows vs. Cuts

- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

# s-t Flow




Function flow: A ➔ R

Flow of arc ≤ arc capacity

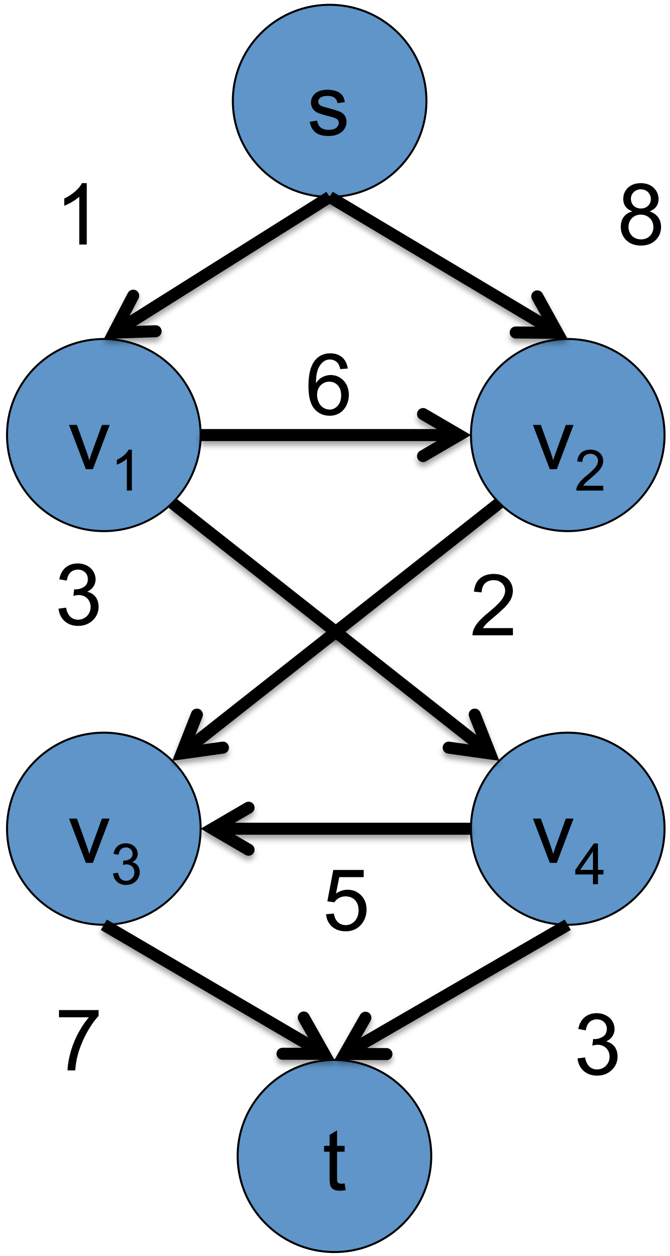
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

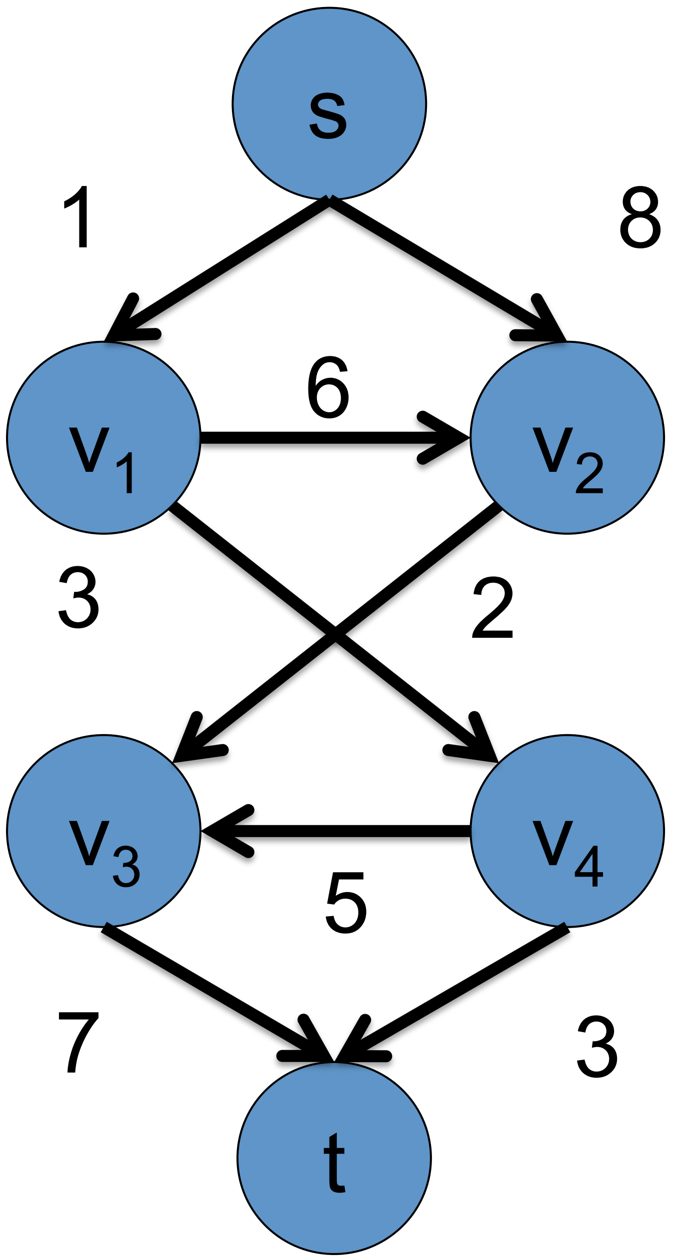
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

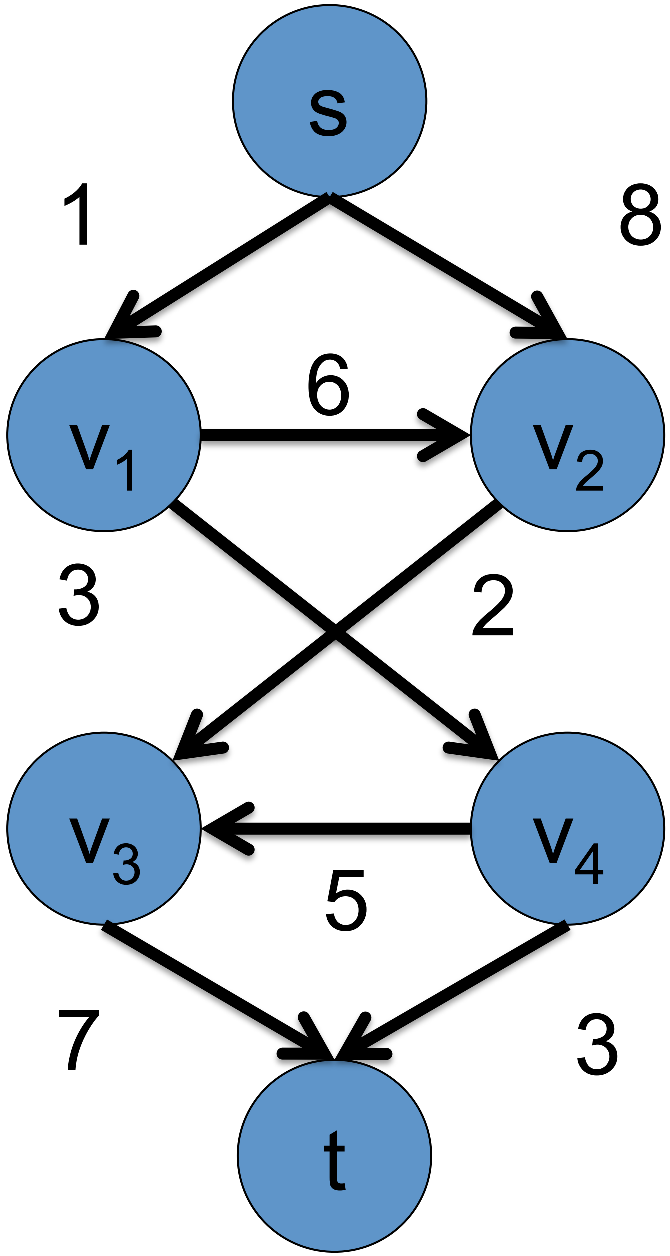
flow(a) ≥ 0

For all vertex except s,t

Incoming flow

= Outgoing flow

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

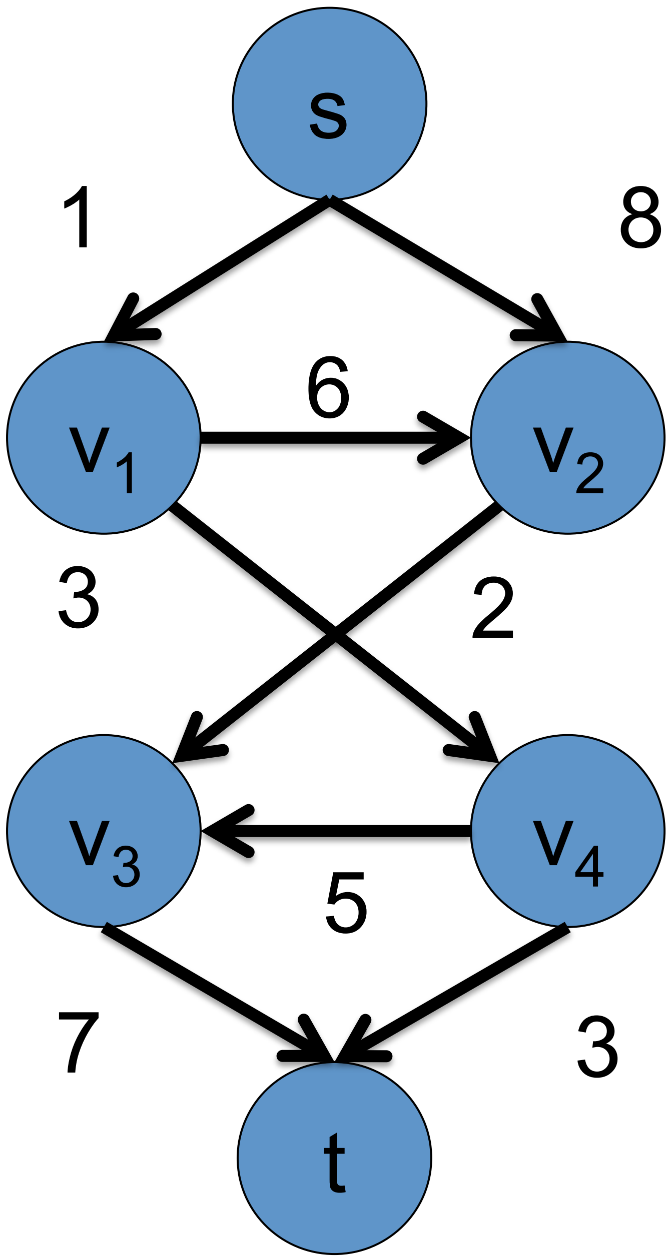
flow(a) ≥ 0

For all v ∈ V \ {s,t}

Incoming flow

= Outgoing flow

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

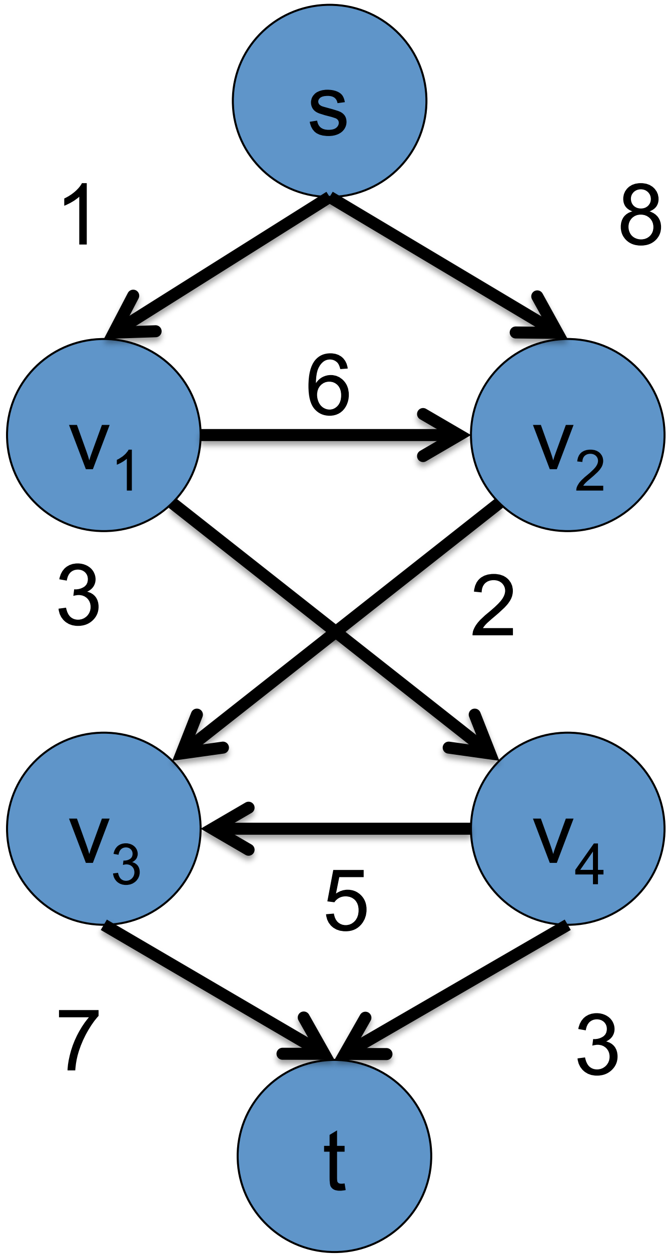
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$\Sigma_{(u,v)\in A}$ flow((u,v))

= Outgoing flow

# s-t Flow




Function flow: A ➔ R

$\text{flow}(a) \leq c(a)$

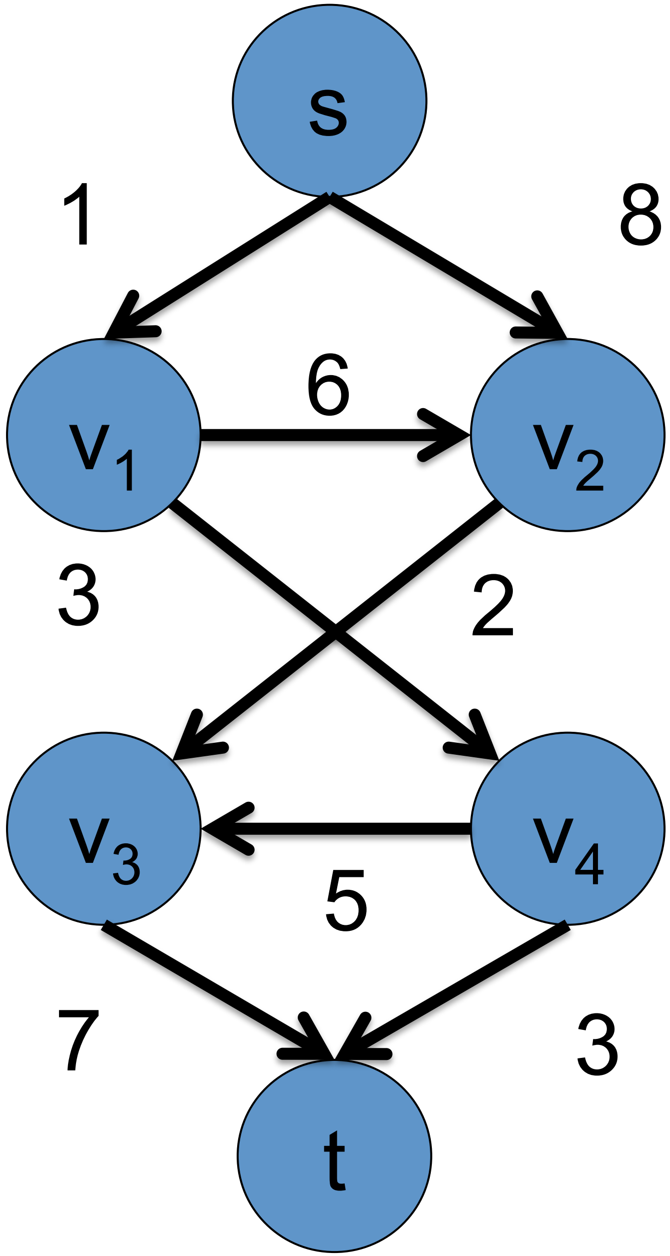
$\text{flow}(a) \geq 0$

For all $v \in V \setminus \{s,t\}$

$$\sum_{(u,v)\in A} \text{flow}((u,v))$$

$$= \sum_{(v,u)\in A} \text{flow}((v,u))$$

# s-t Flow




Function flow: A ➔ R

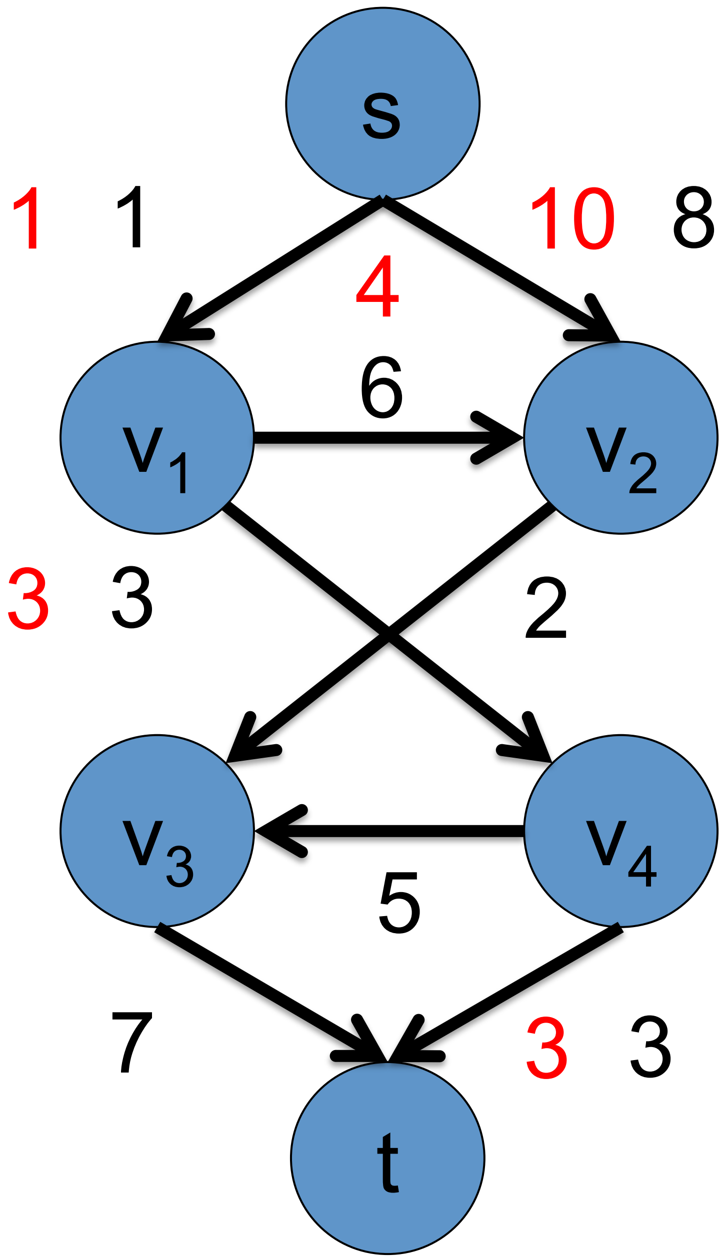
flow(a) ≤ c(a)

flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

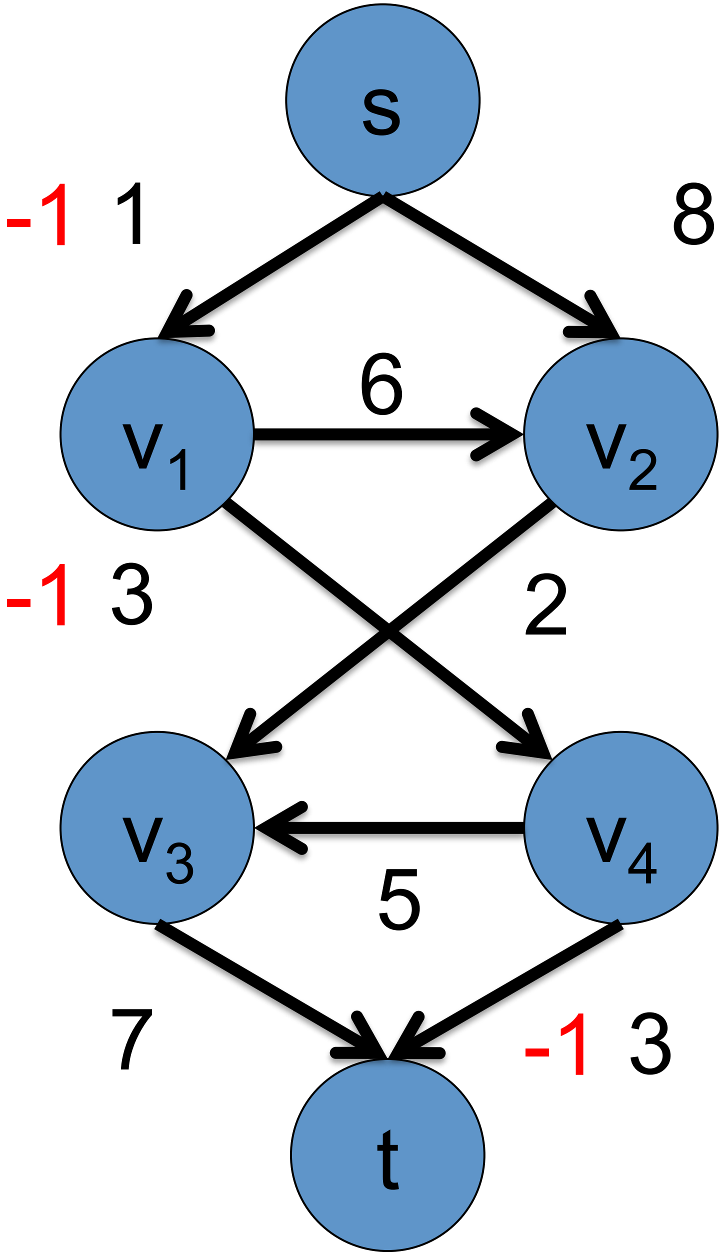
flow(a) ≥ 0

For all v ∈ V \ {s,t}

$E_{flow}(v) = 0$

✗

# s-t Flow




Function flow: A → R

flow(a) ≤ c(a)

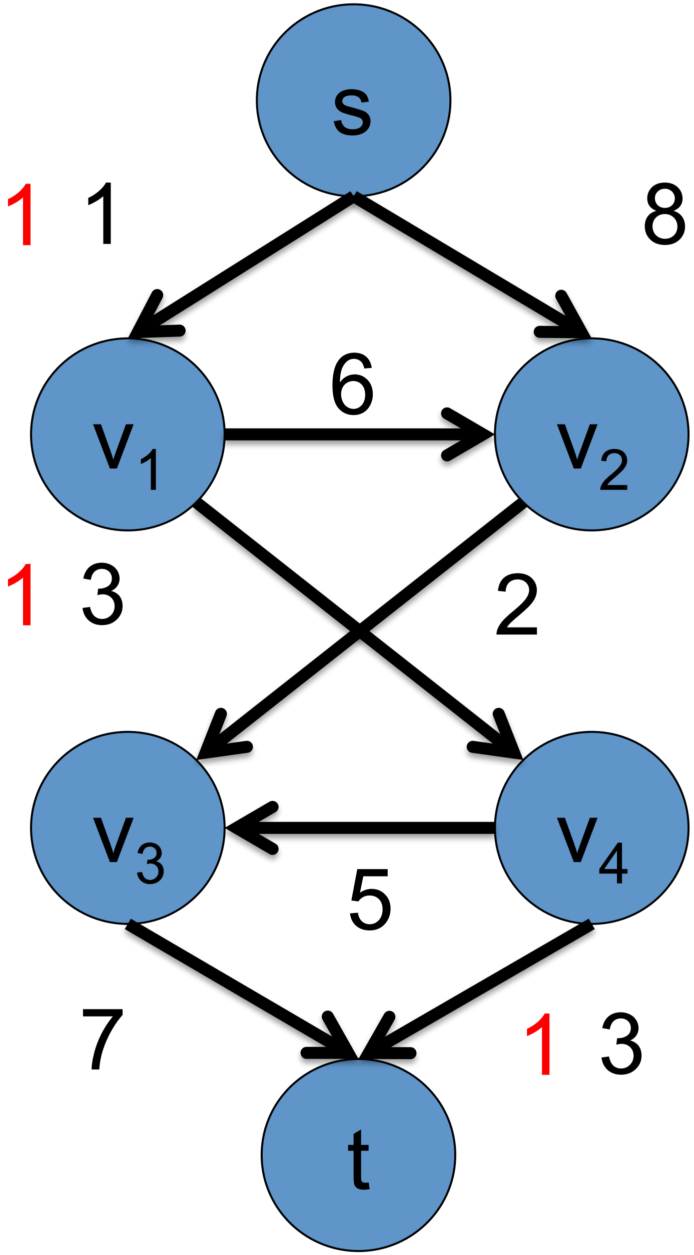
flow(a) ≥ 0

For all v ∈ V \ {s,t}

$E_{flow}(v) = 0$

✗

# s-t Flow




Function flow: A ➔ R

flow(a) ≤ c(a)

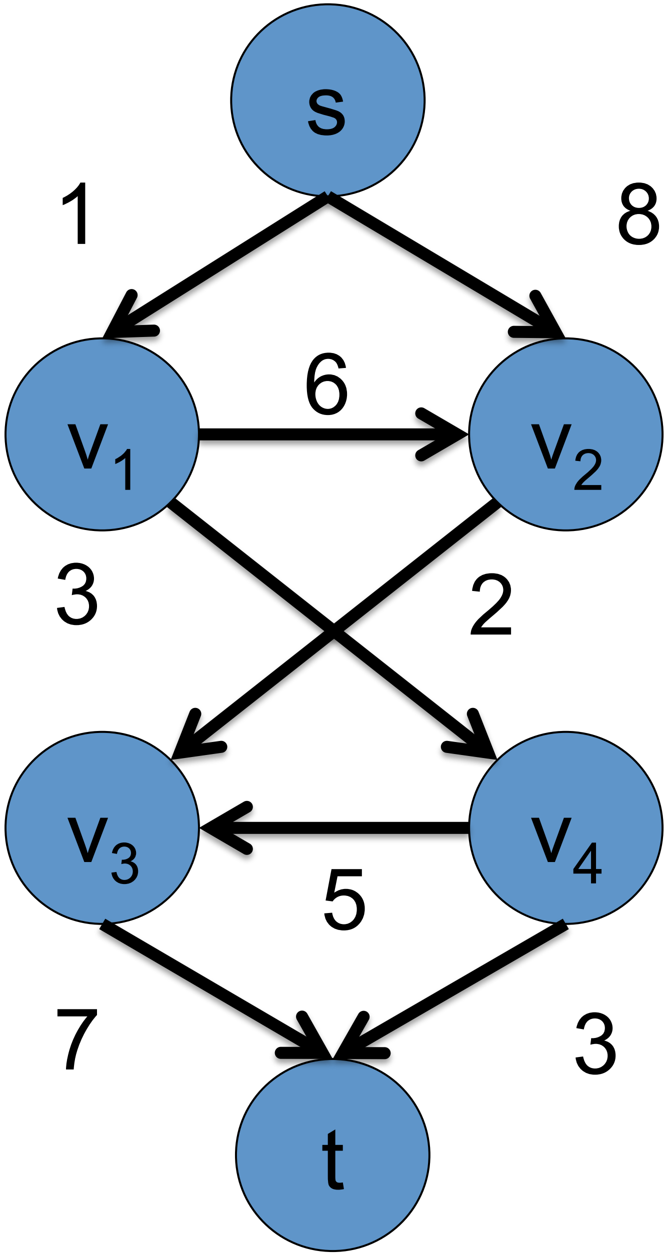
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

✓

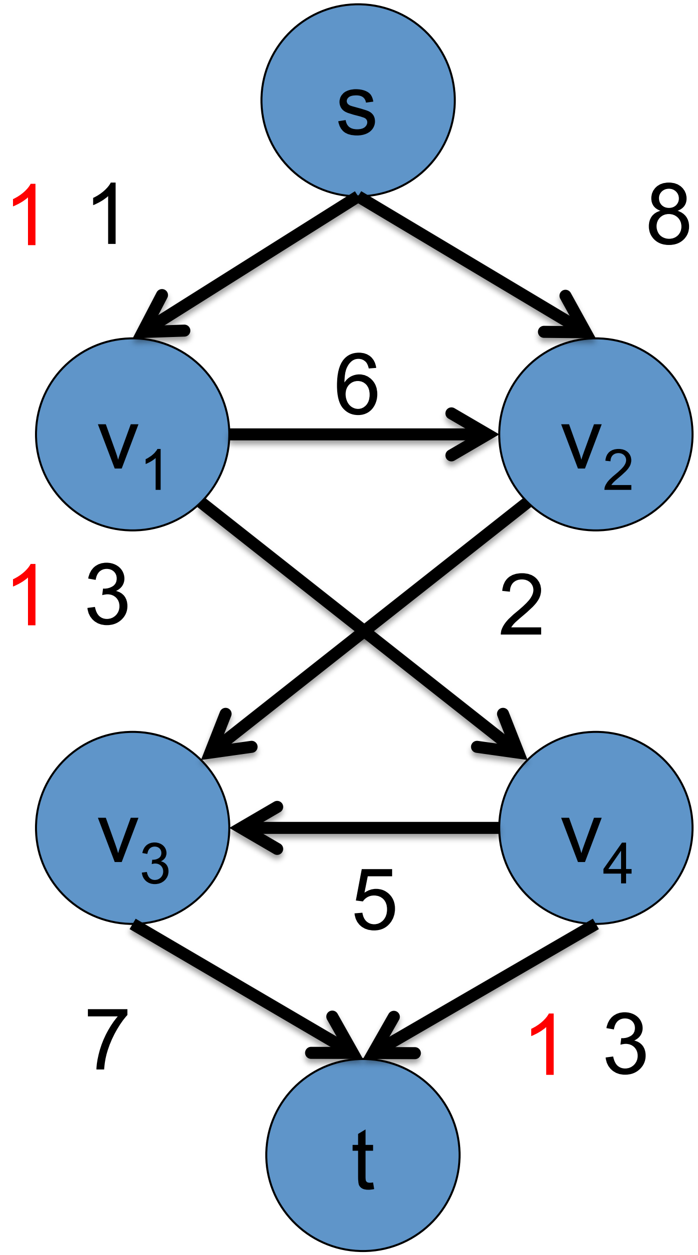
# Value of s-t Flow




Outgoing flow of s

- Incoming flow of s

# Value of s-t Flow




$-E_{flow}(s)$ $E_{flow}(t)$

$$\Sigma_{(s,v)\in A} \text{ flow}((s,v))$$
$$- \Sigma_{(u,s)\in A} \text{ flow}((u,s))$$

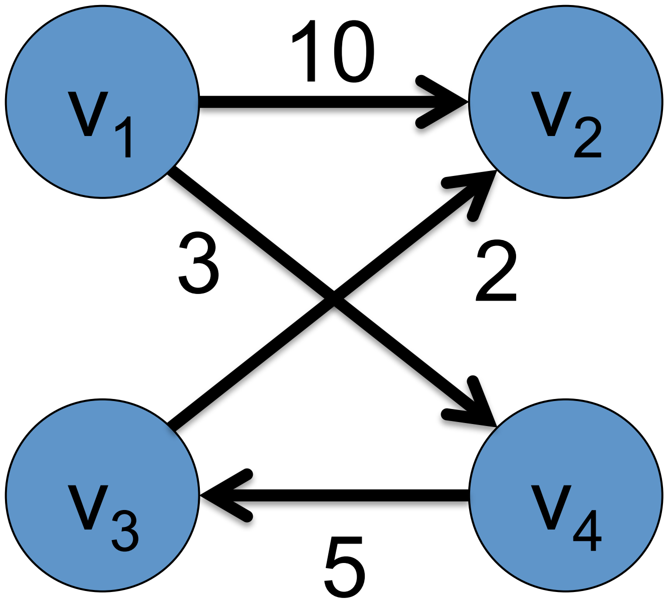
Value = 1

# Outline

- Preliminaries
  - Functions and Excess Functions
  - s-t Flow
  - **s-t Cut**
  - Flows vs. Cuts
- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

# Cut

**D = (V, A)**



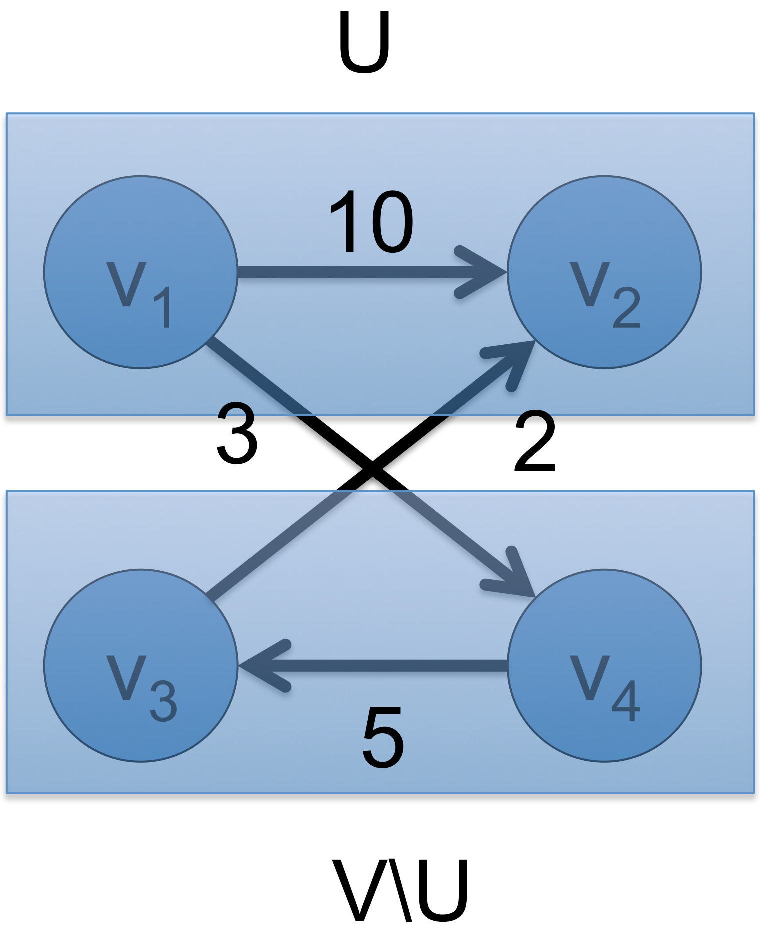

Let U be a subset of V

C is a set of arcs such that

- $(u,v) \in A$
- $u \in U$
- $v \in V \backslash U$

C is a cut in the digraph D

# Cut




**D = (V, A)**

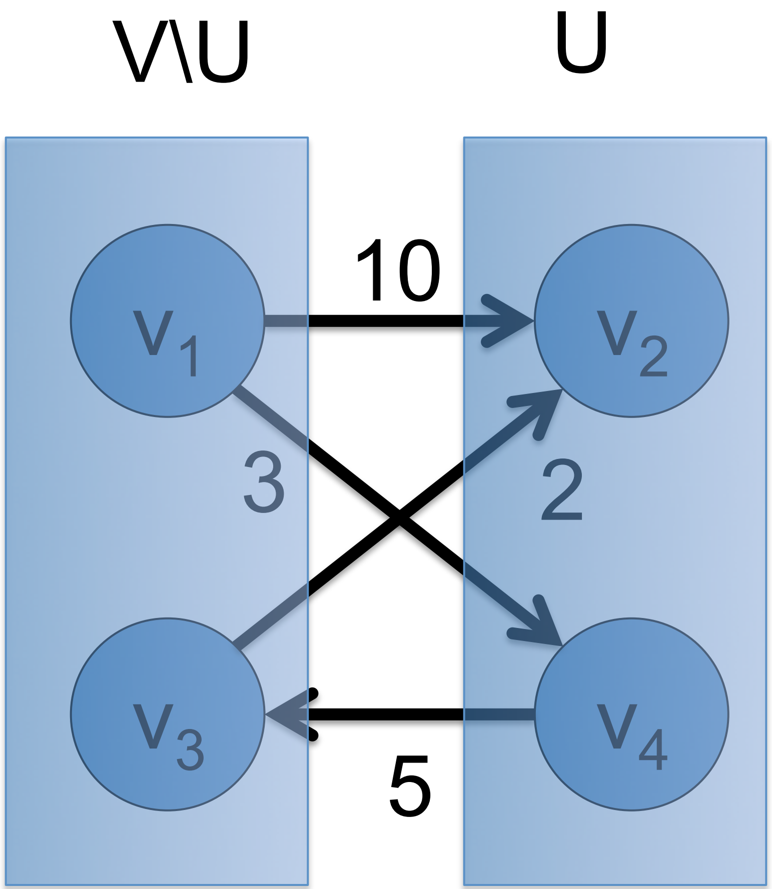
What is C?

$\{(v_1,v_2),(v_1,v_4)\}$ ?

$\{(v_1,v_4),(v_3,v_2)\}$ ?

✓ $\{(v_1,v_4)\}$ ?

# Cut




**D = (V, A)**

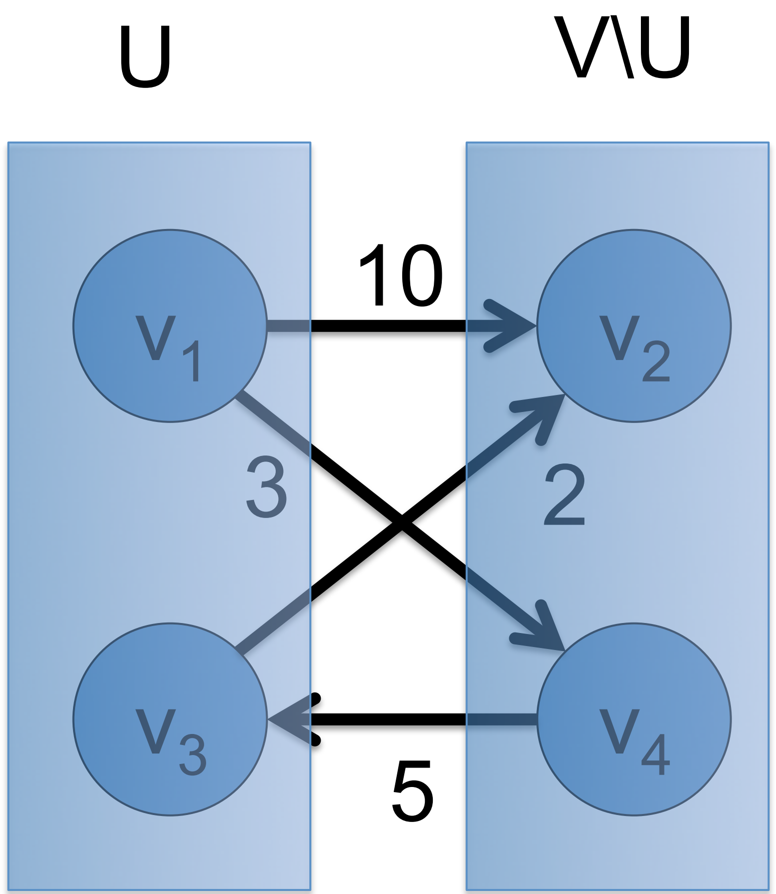
What is C?

$\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$ ?

✓ $\{(v_4,v_3)\}$ ?

$\{(v_1,v_4),(v_3,v_2)\}$ ?

# Cut




**$D = (V, A)$**

What is C?

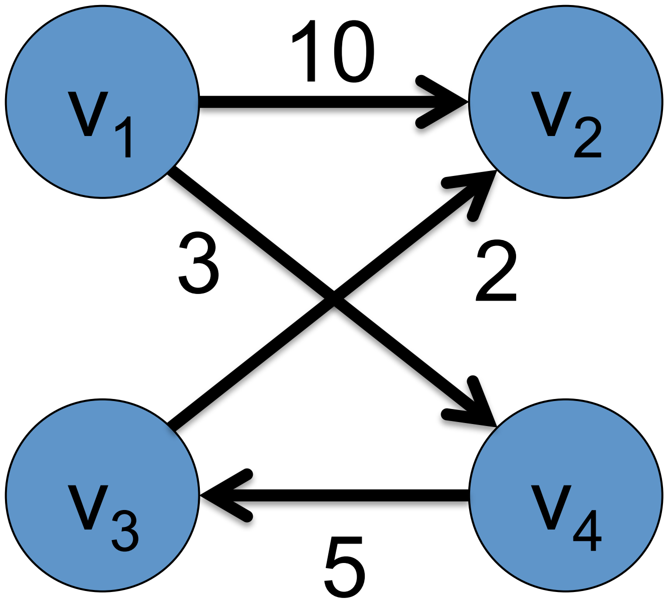
✓ $\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$ ?

$\{(v_3,v_2)\}$ ?

$\{(v_1,v_4),(v_3,v_2)\}$ ?

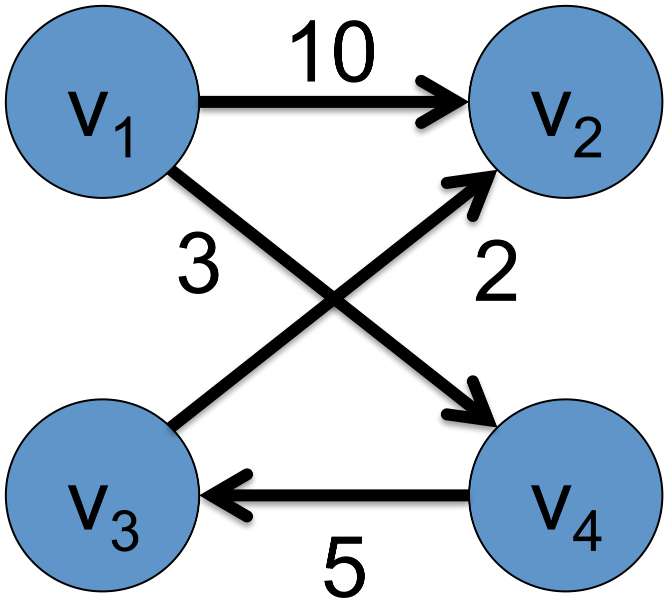
# Cut

**D = (V, A)**



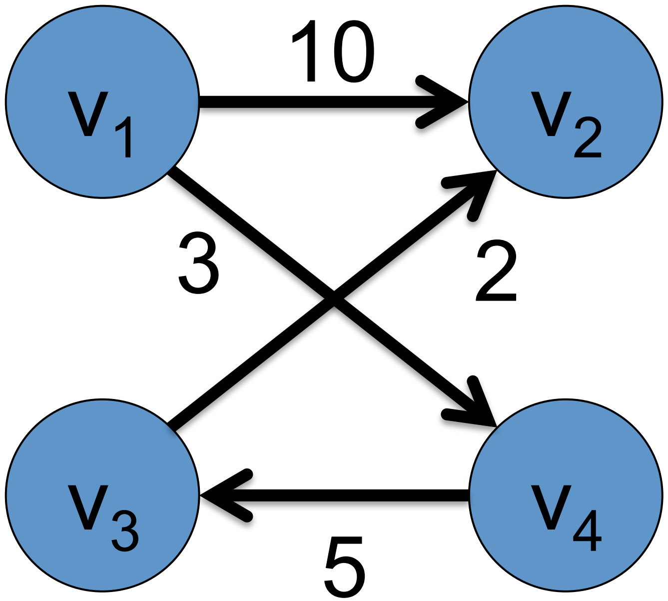

C = out-arcs(U)

# Capacity of Cut



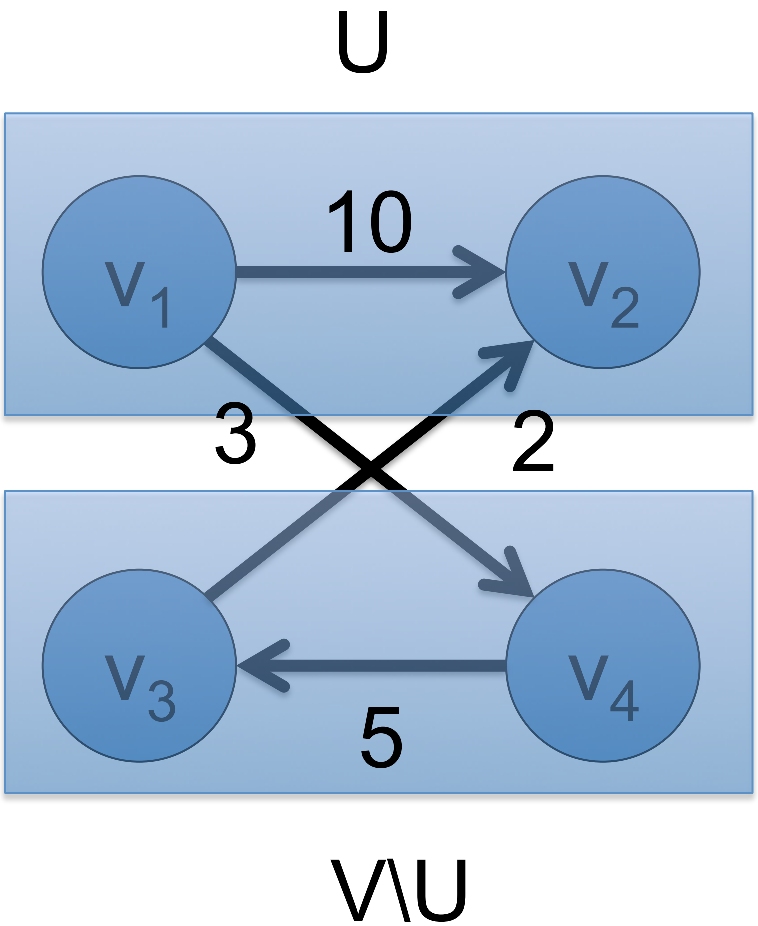

Sum of capacity of all arcs in C

# Capacity of Cut




$\Sigma_{a \in C}\ c(a)$

# Capacity of Cut

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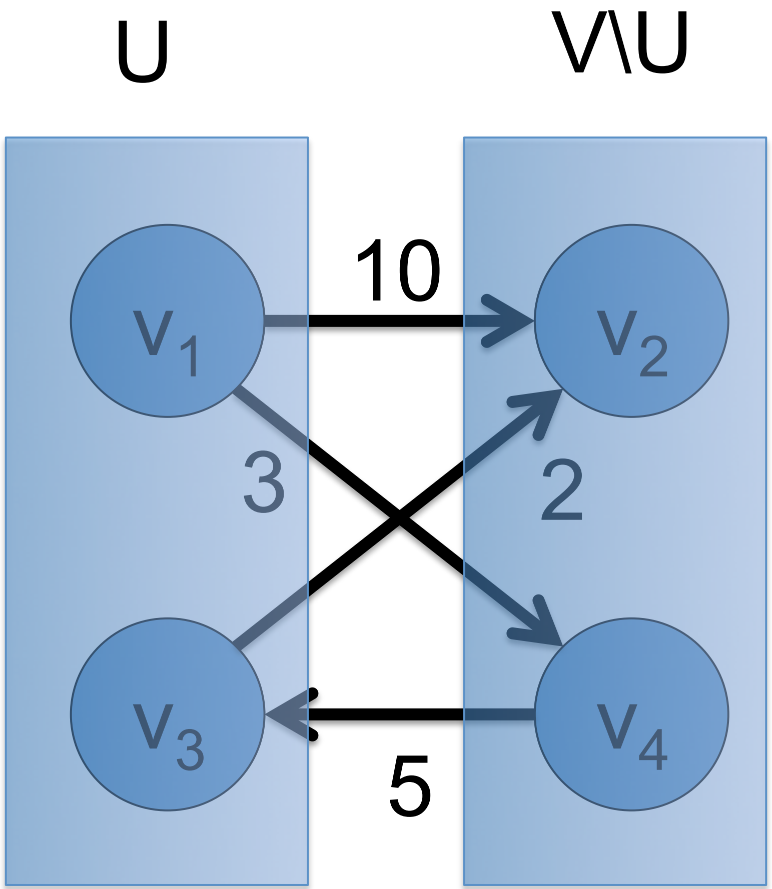
# Capacity of Cut

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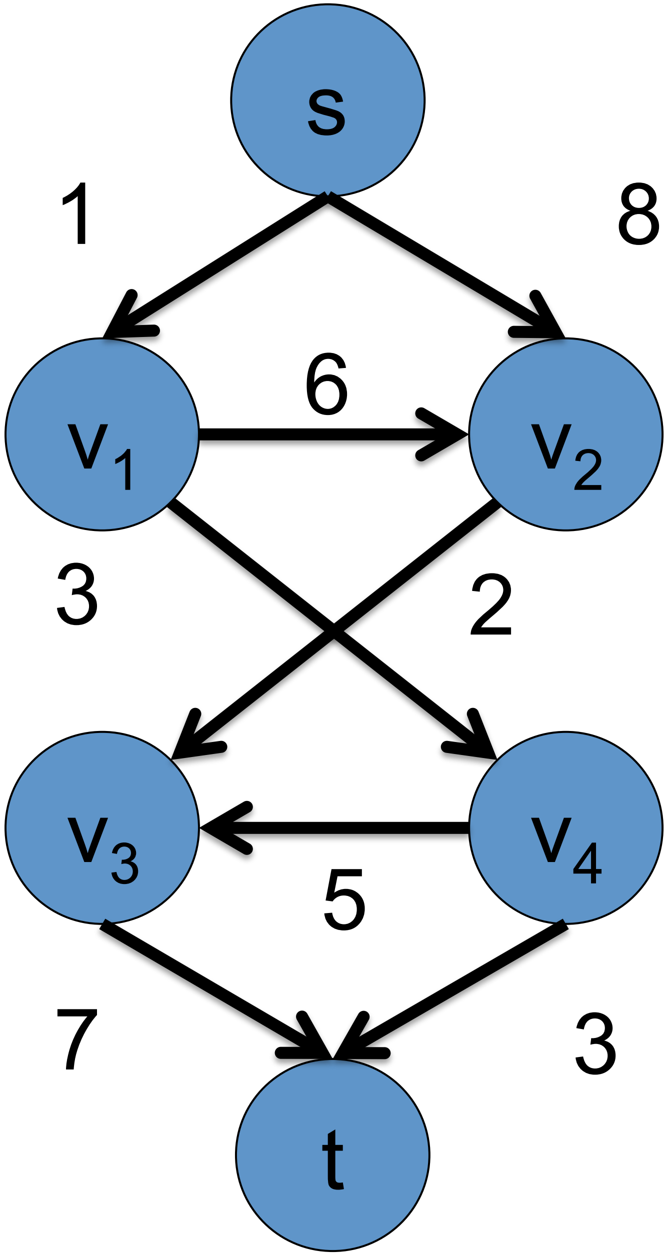
# s-t Cut




**D = (V, A)**

A source vertex "s"

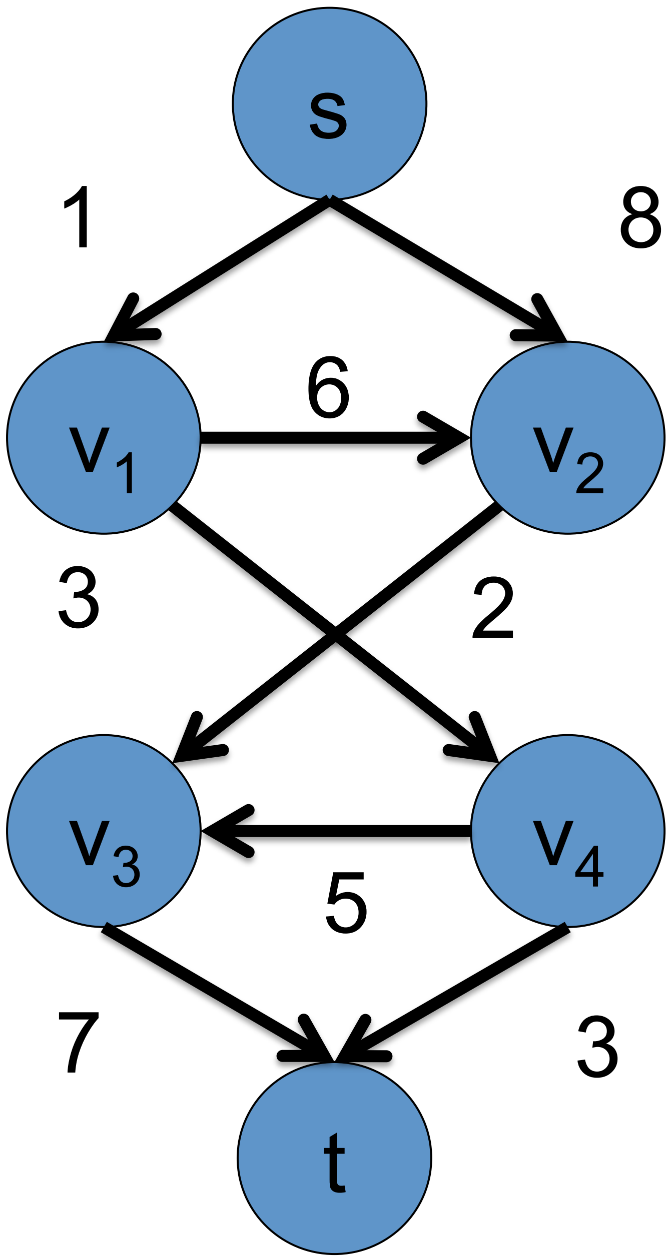
A sink vertex "t"

C is a cut such that

- $s \in U$
- $t \in V \backslash U$

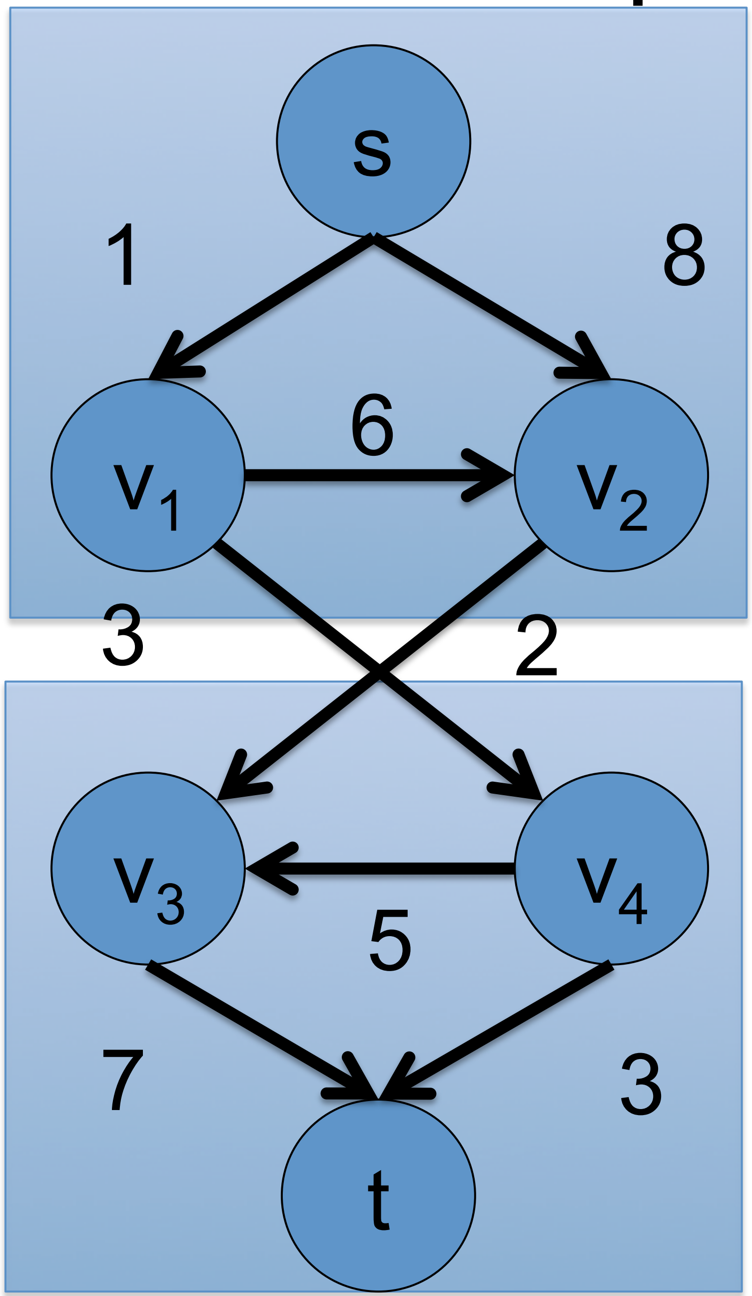
C is an s-t cut

# Capacity of s-t Cut




$\Sigma_{a \in C}\ c(a)$

# Capacity of s-t Cut

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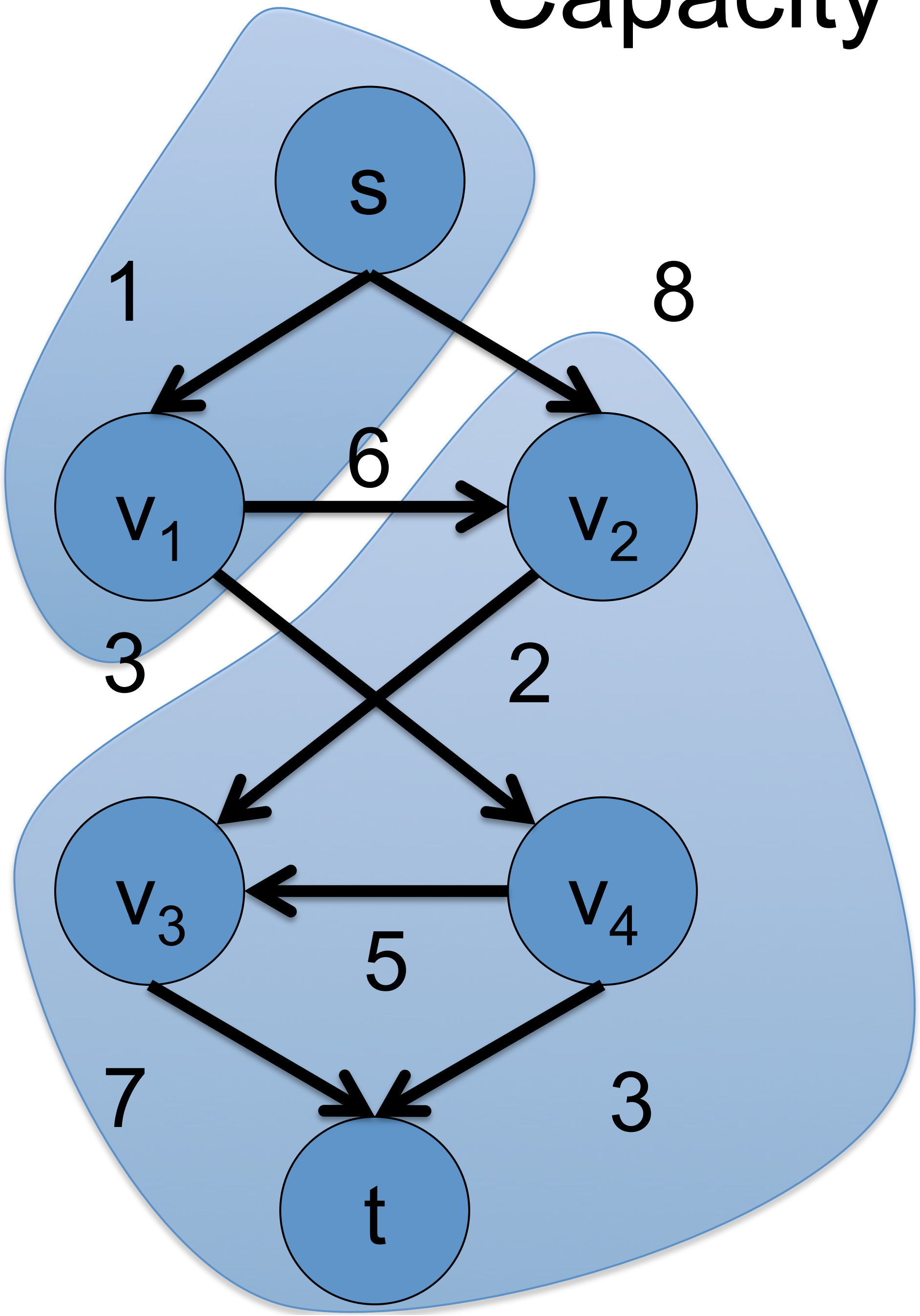
# Capacity of s-t Cut

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# Outline

- Preliminaries
  - s-t Flow
  - s-t Cut
  - **Flows vs. Cuts**
- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

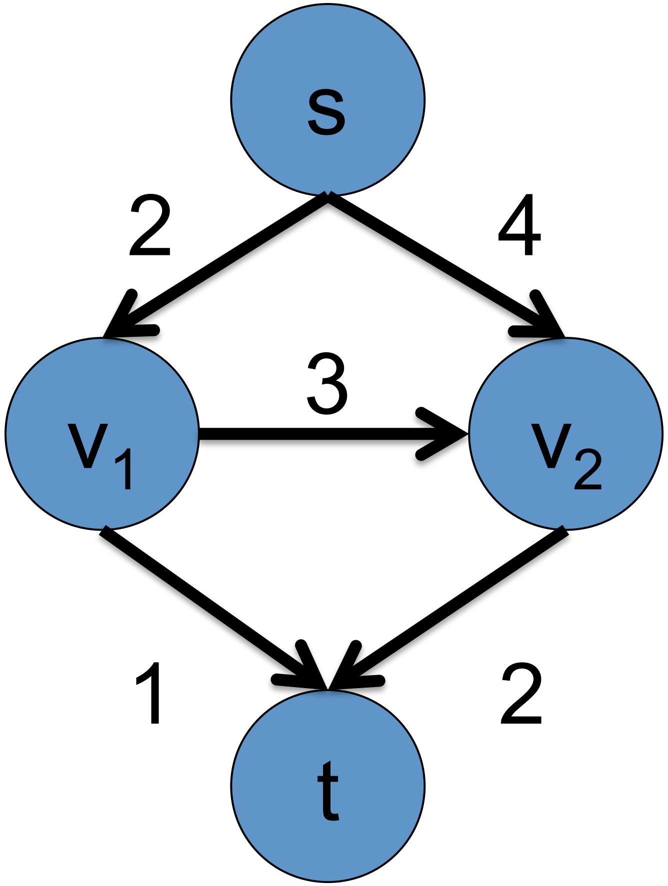
# Flows vs. Cuts

(See lecture 5)

# Outline

- Preliminaries
- **Maximum Flow**
  - Residual Graph
  - Max-Flow Min-Cut Theorem
- Algorithms
- Energy minimization with max flow/min cut

# Maximum Flow Problem



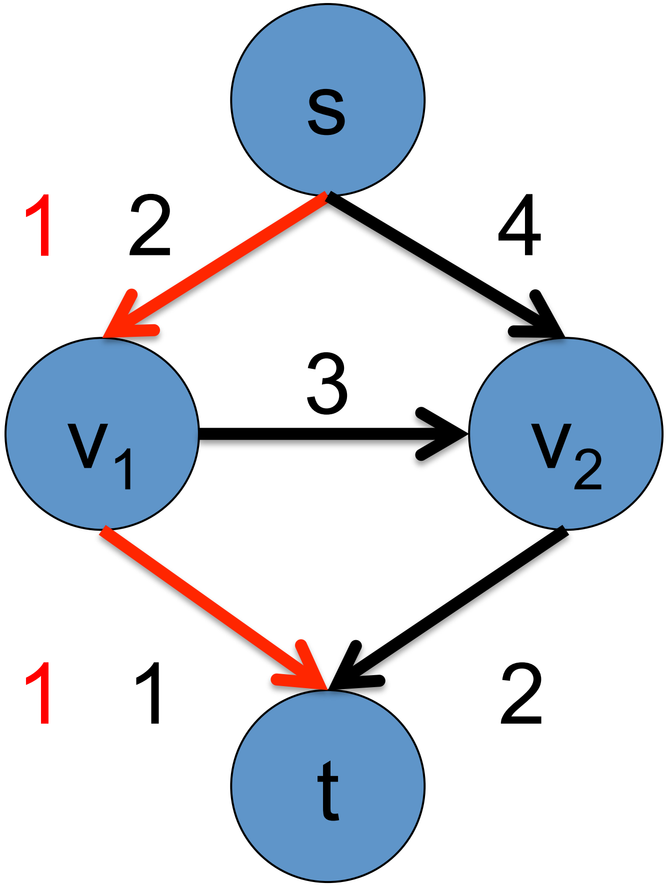

Find the flow with the maximum value !!

$$\Sigma_{(s,v)\in A} \text{flow}((s,v))$$

$$- \Sigma_{(u,s)\in A} \text{flow}((u,s))$$

**First suggestion to solve this problem !!**

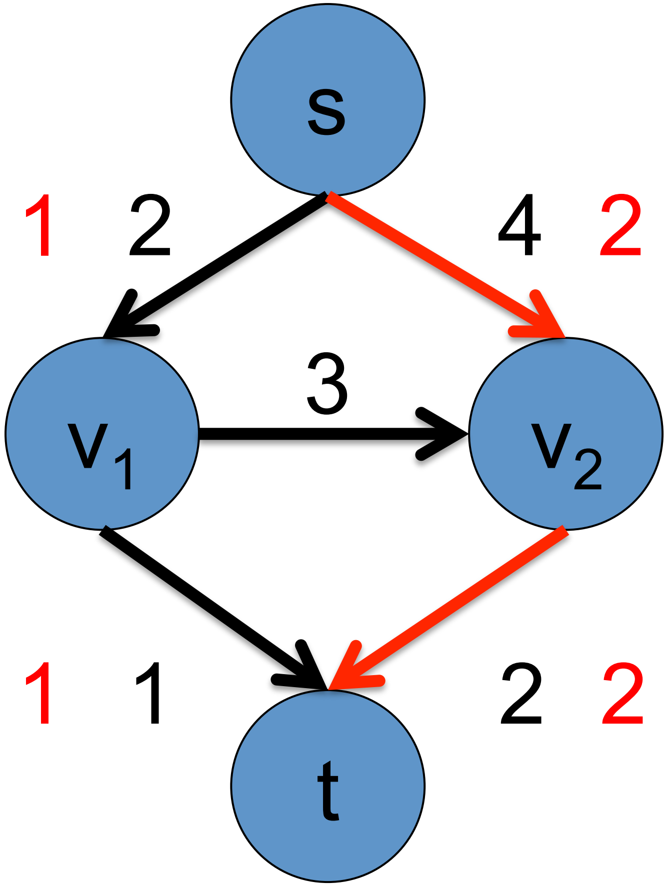
# Passing Flow through s-t Paths




Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

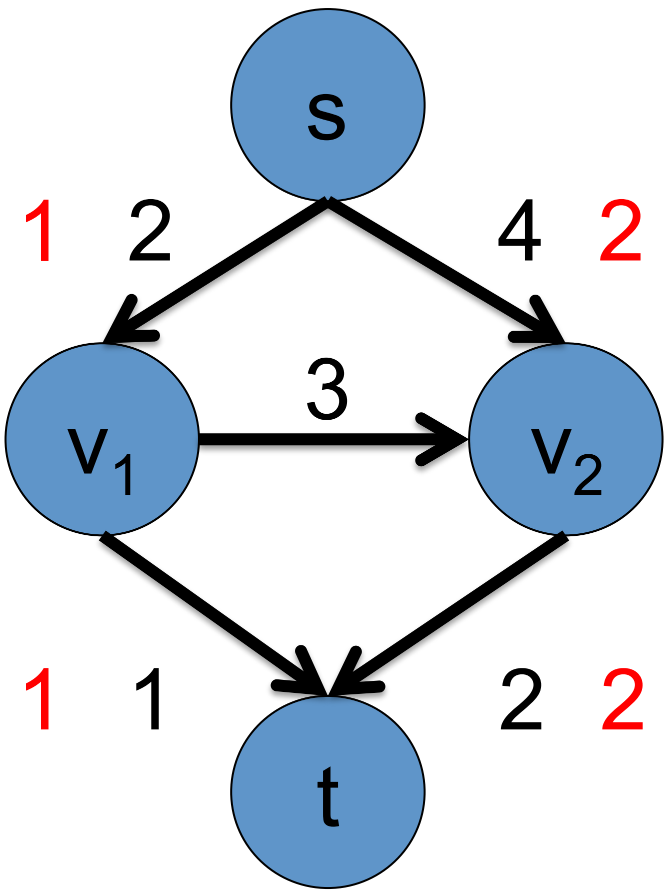
# Passing Flow through s-t Paths




Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

# Passing Flow through s-t Paths

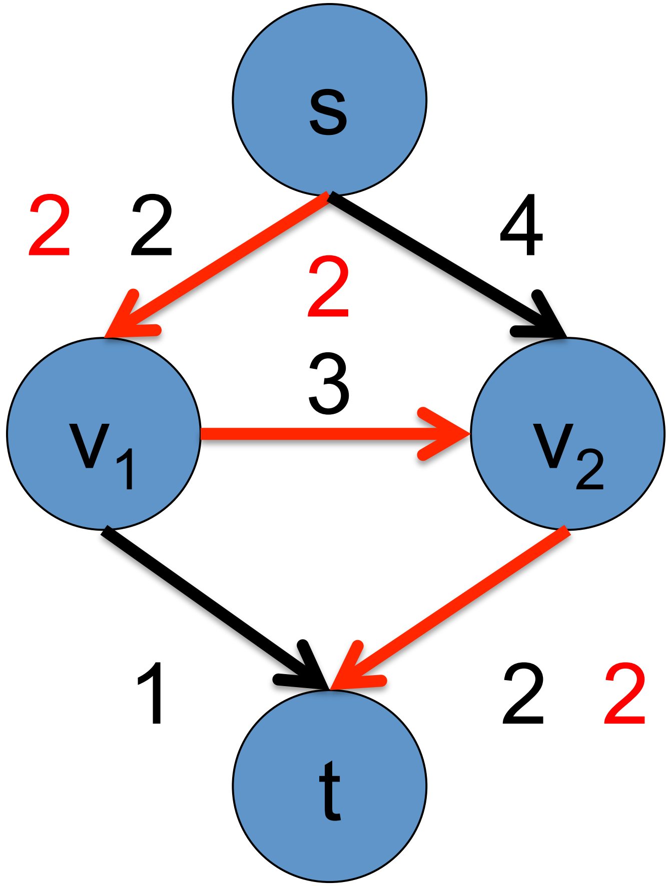



Find an s-t path where flow(a) < c(a) for all arcs

No more paths. Stop.

Will this give us maximum flow? **NO !!!**

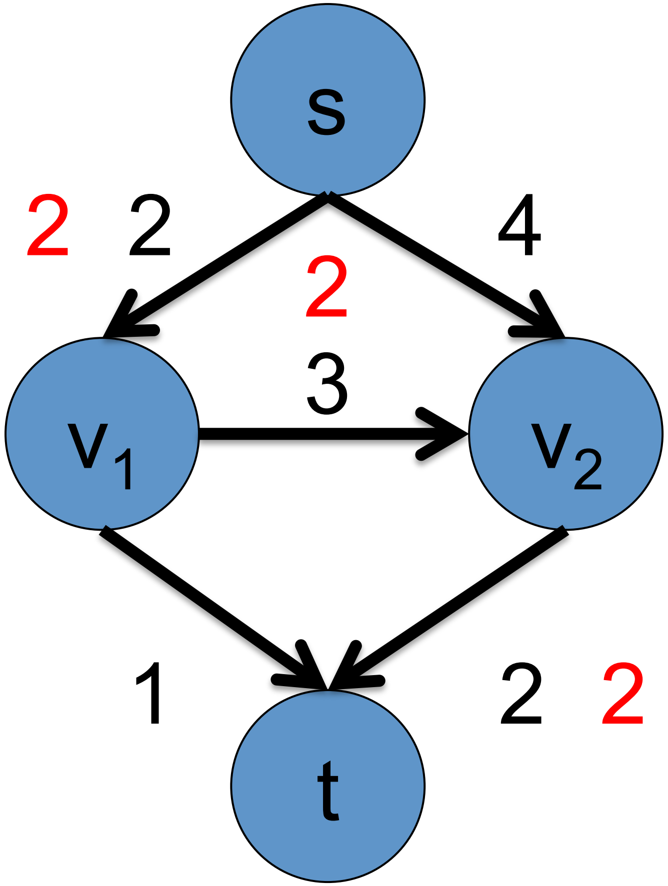
# Passing Flow through s-t Paths




Find an s-t path where flow(a) < c(a) for all arcs

Pass maximum allowable flow through the arcs

# Passing Flow through s-t Paths




Find an s-t path where flow(a) < c(a) for all arcs

No more paths. Stop.

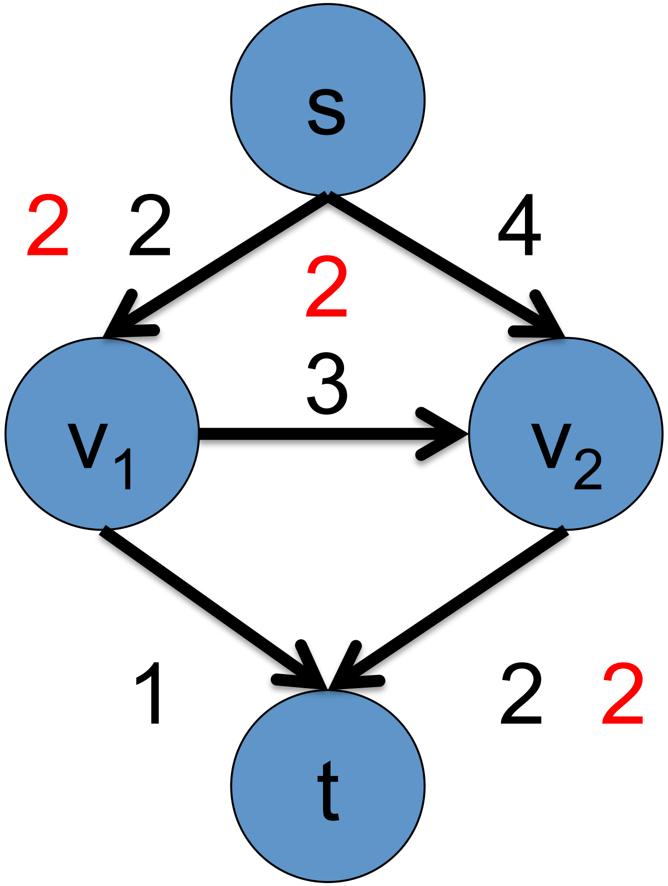
**Another method?**

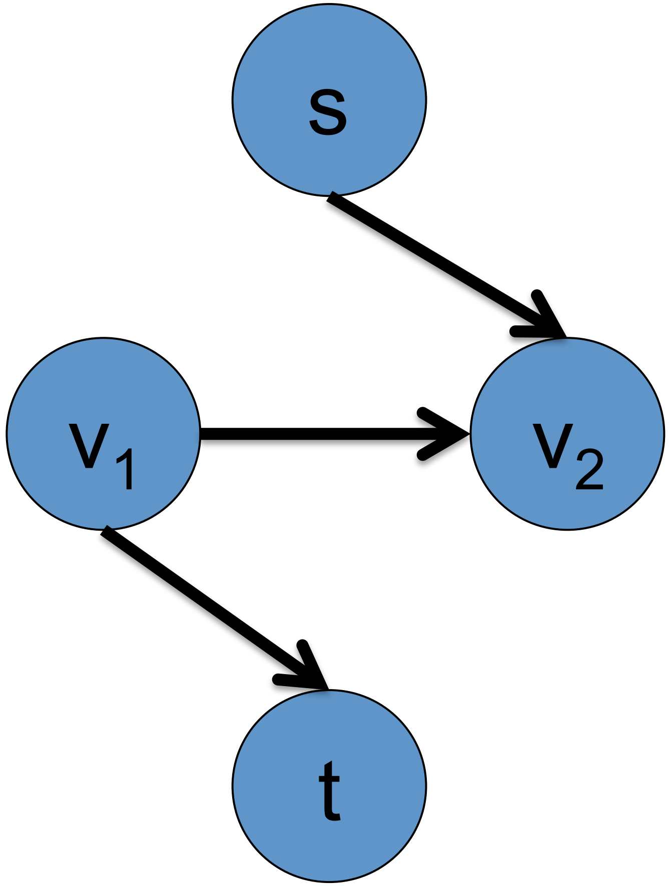
Incorrect Answer !!

# Outline

- Preliminaries
- Maximum Flow
  - **Residual Graph**
  - Max-Flow Min-Cut Theorem
- Algorithms
- Energy minimization with max flow/min cut

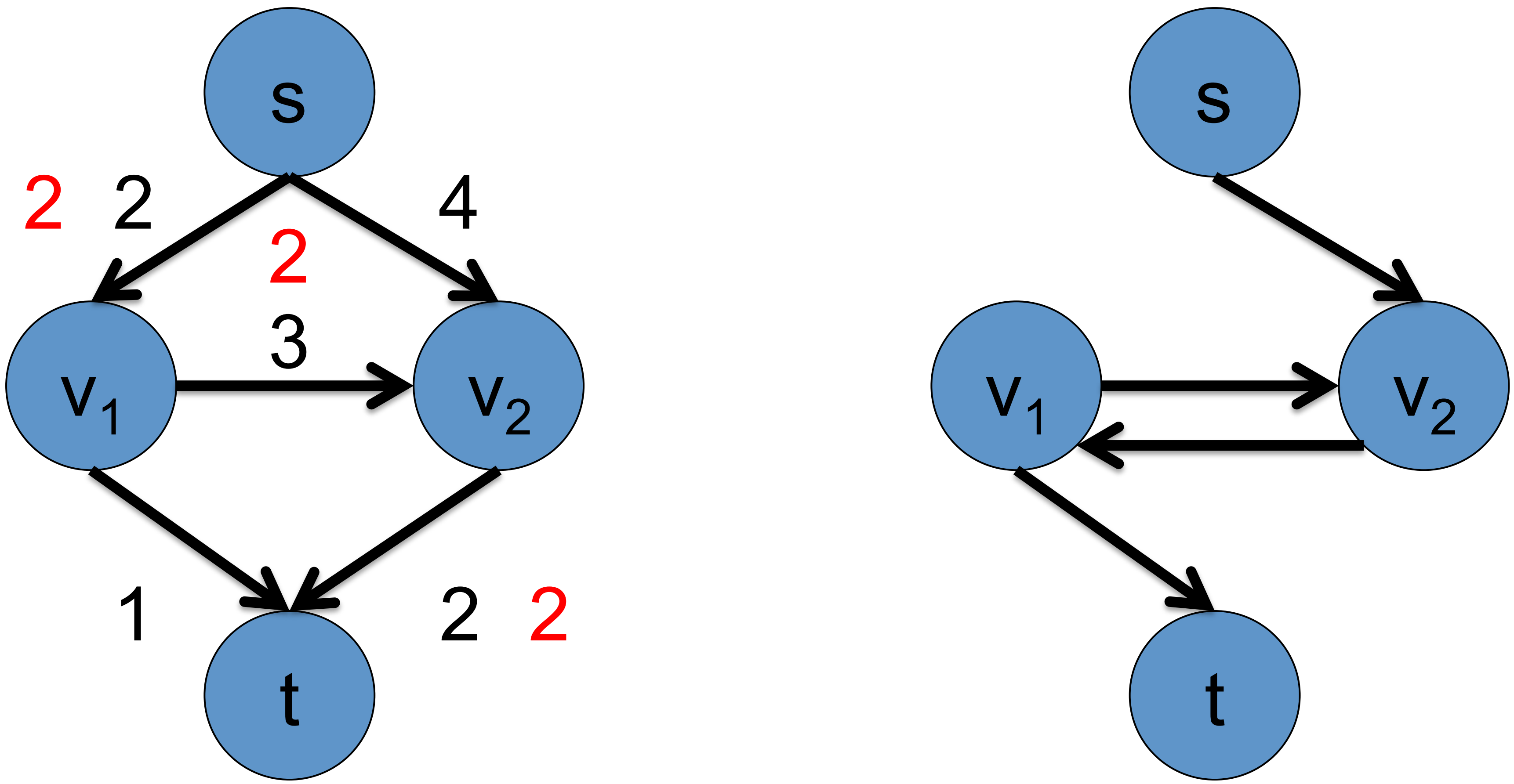
# Residual Graph




Arcs where flow(a) < c(a)

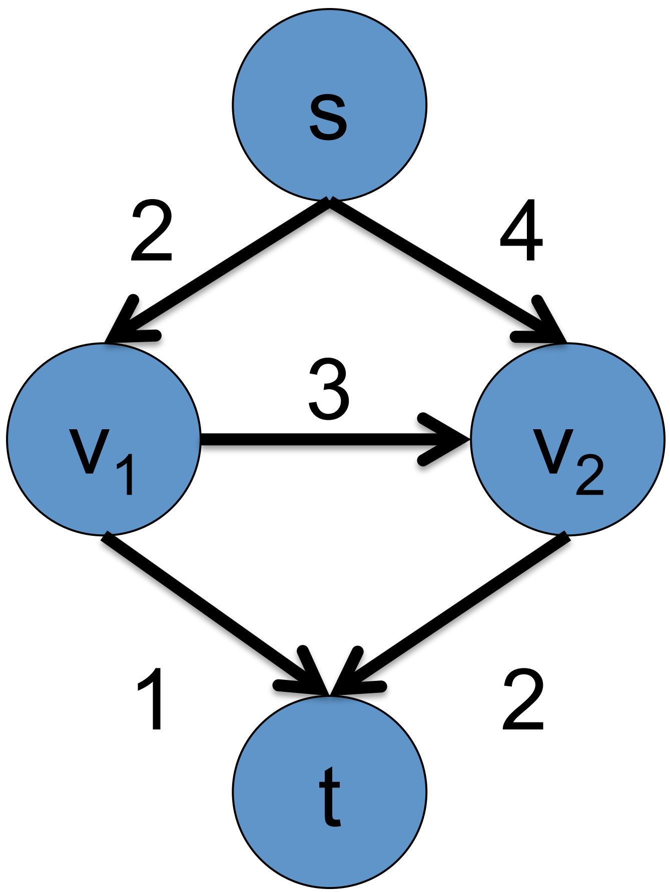
# Residual Graph

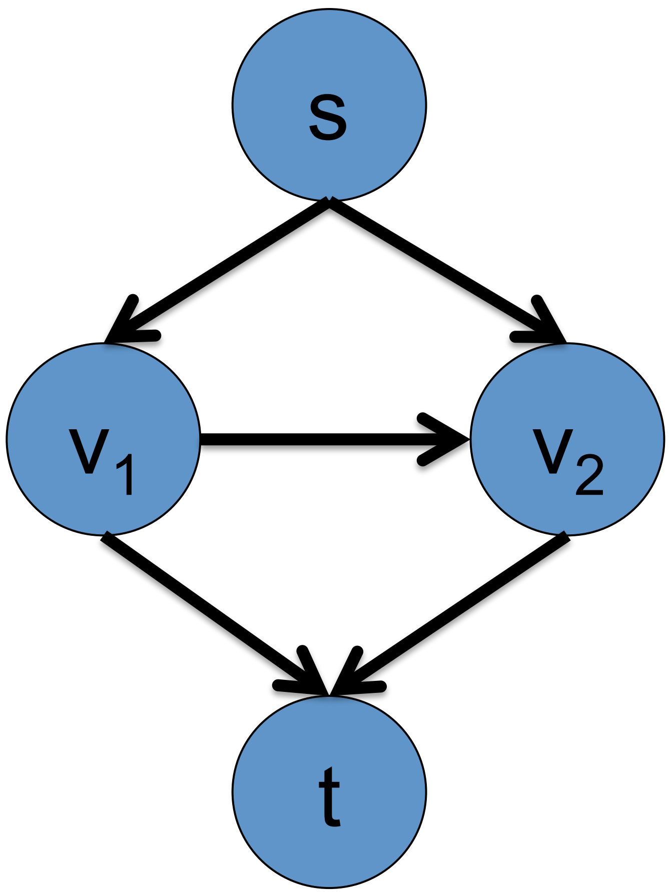



Including arcs to s and from t is not necessary

Inverse of arcs where flow(a) > 0

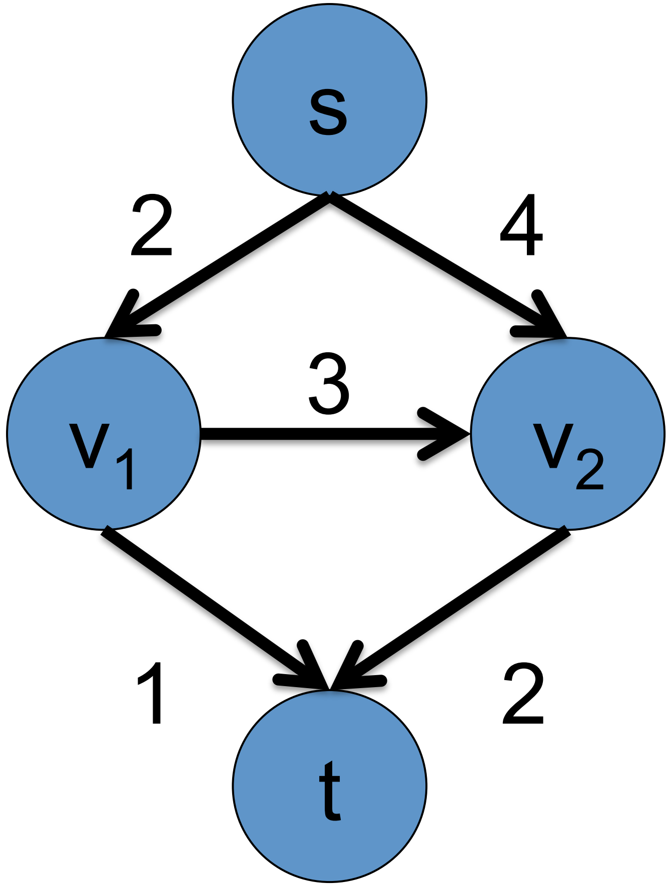
# Maximum Flow using Residual Graphs

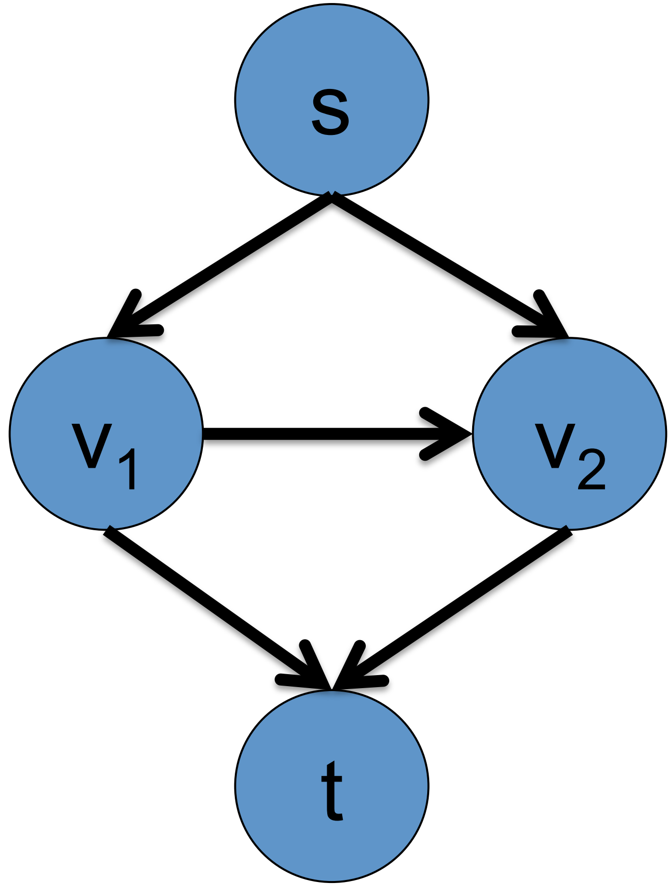



Start with zero flow.

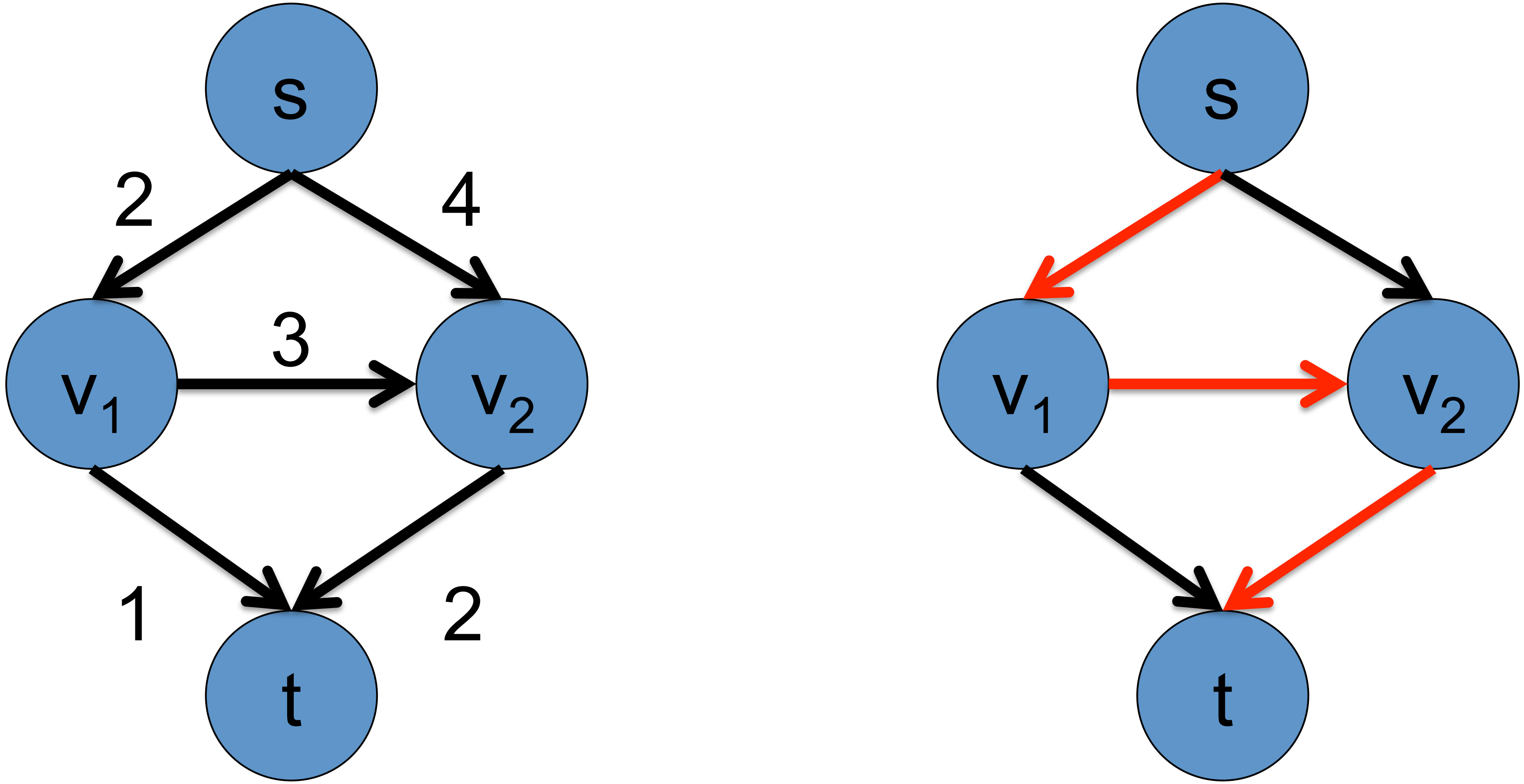
# Maximum Flow using Residual Graphs



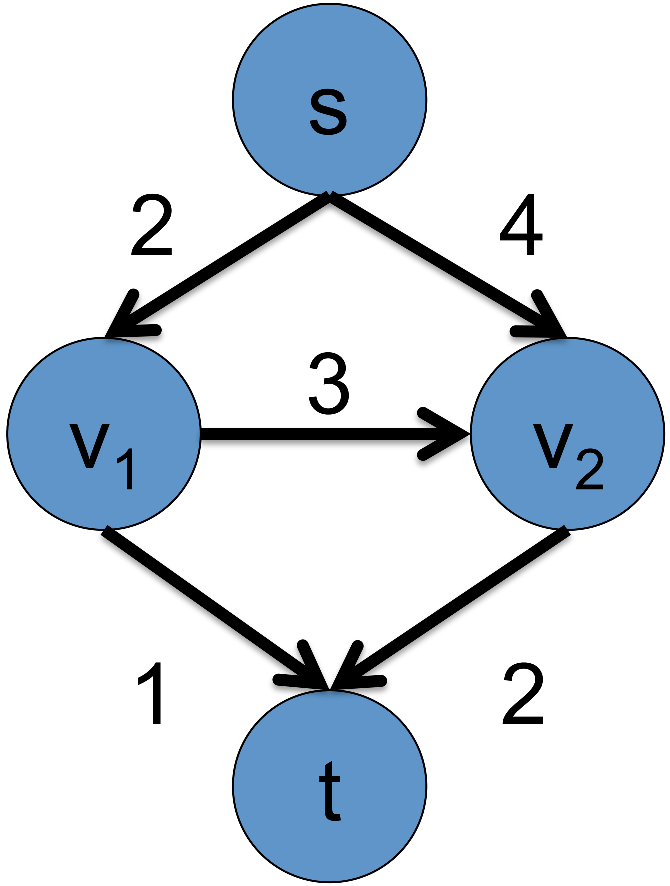

Find an s-t path in the residual graph.

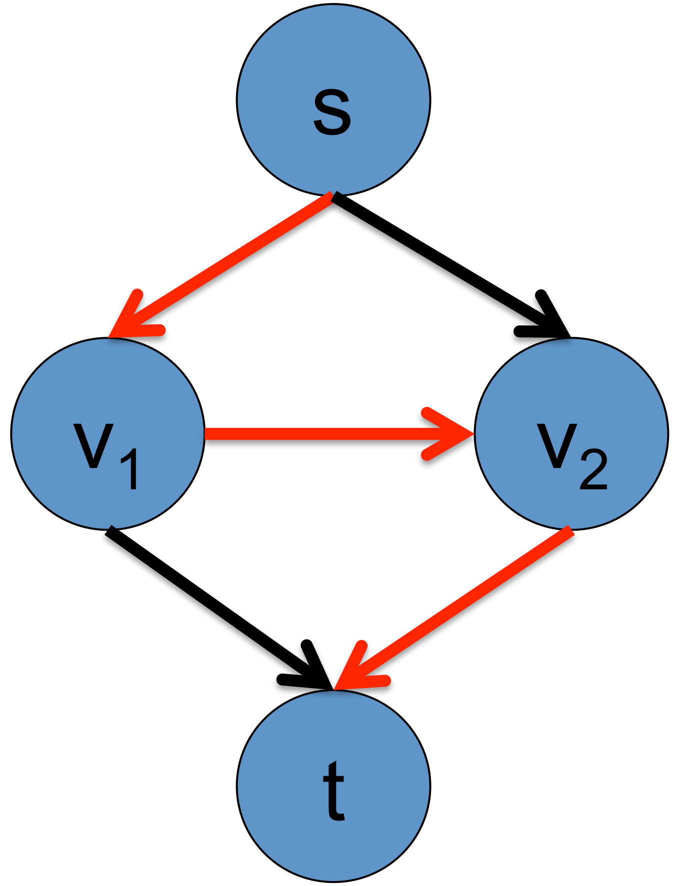
# Maximum Flow using Residual Graphs




Find an s-t path in the residual graph.

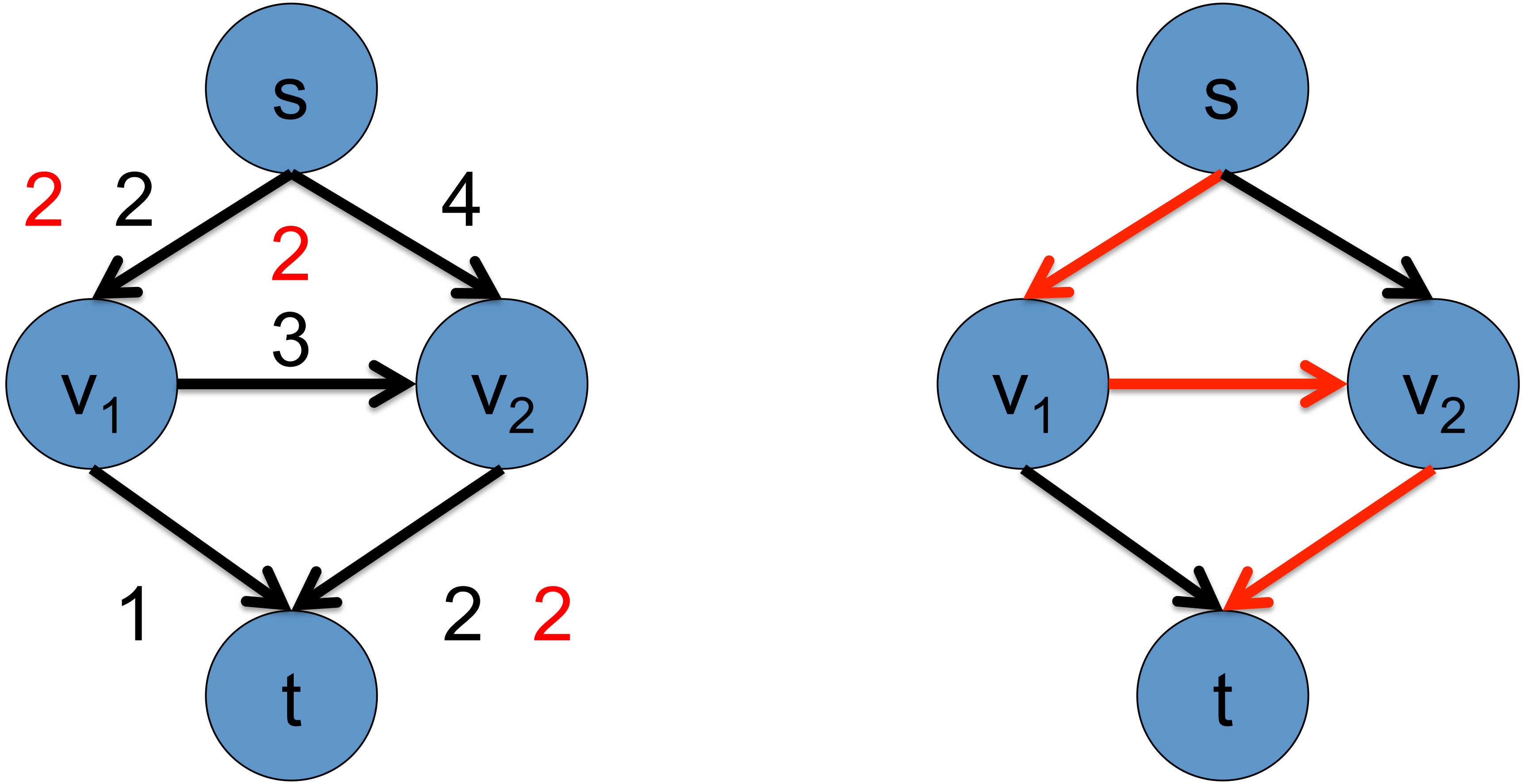
# Maximum Flow using Residual Graphs




For inverse arcs in path, subtract flow K.

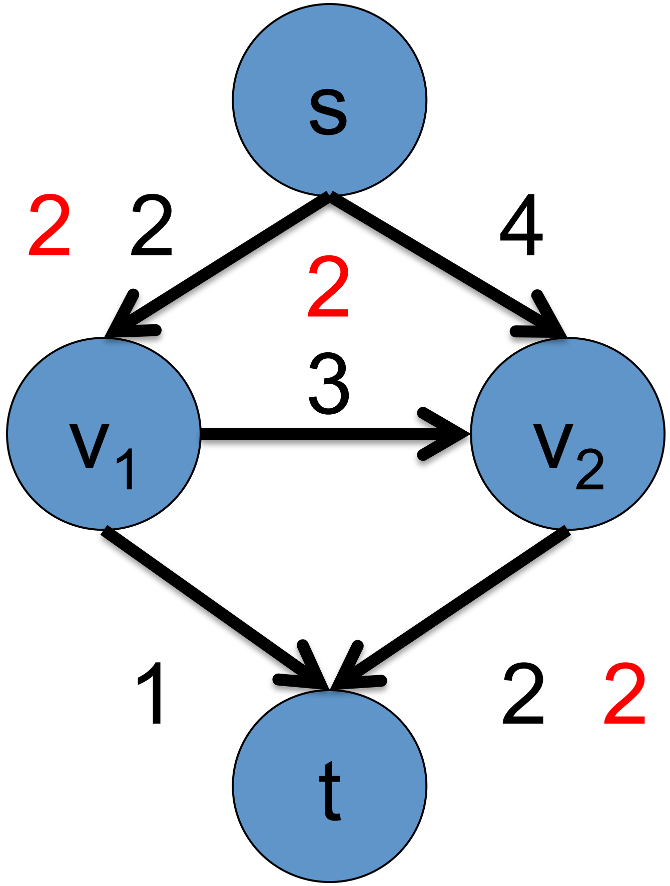
# Maximum Flow using Residual Graphs

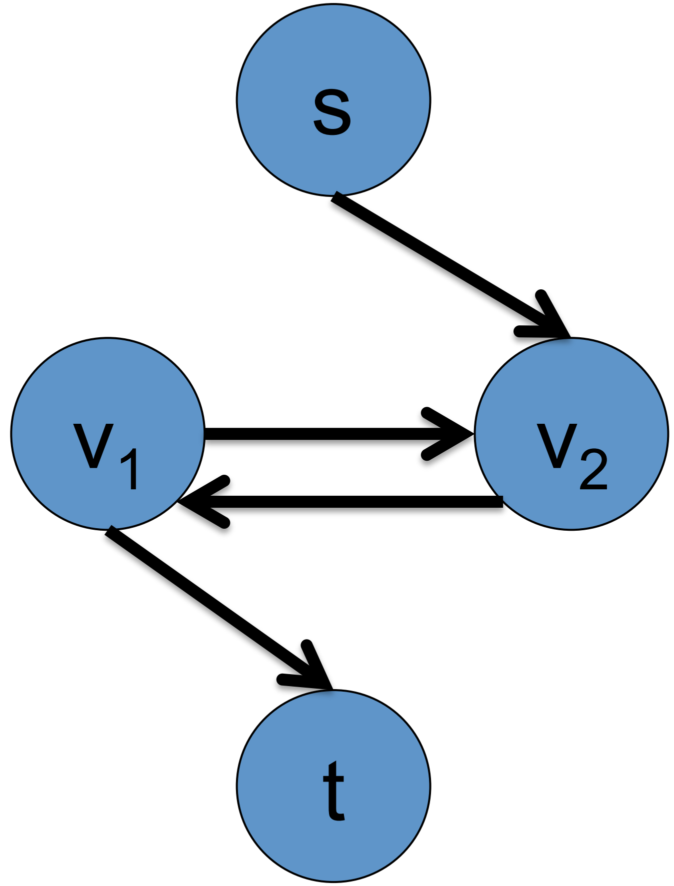



Choose maximum allowable value of K.

For forward arcs in path, add flow K.

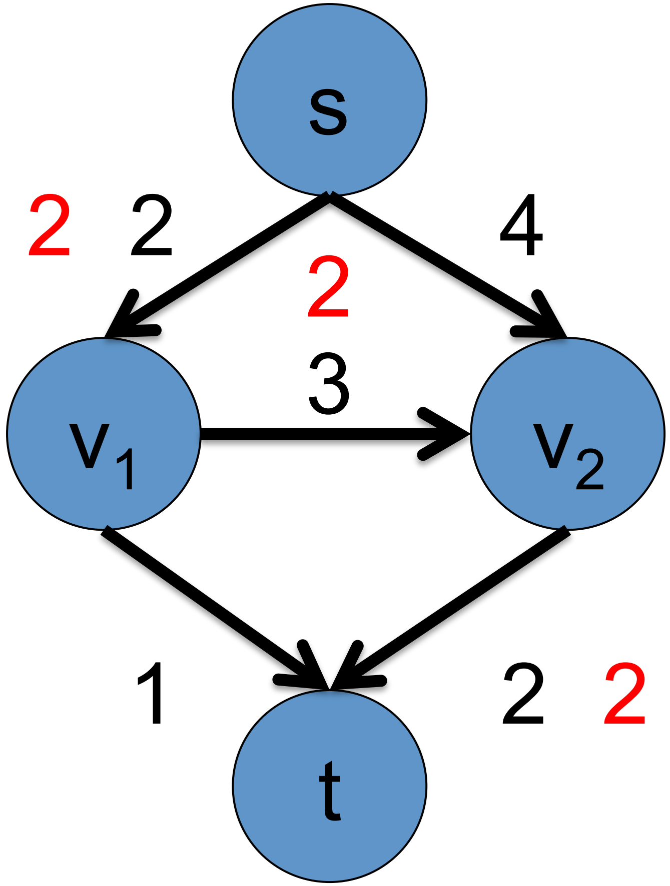
# Maximum Flow using Residual Graphs

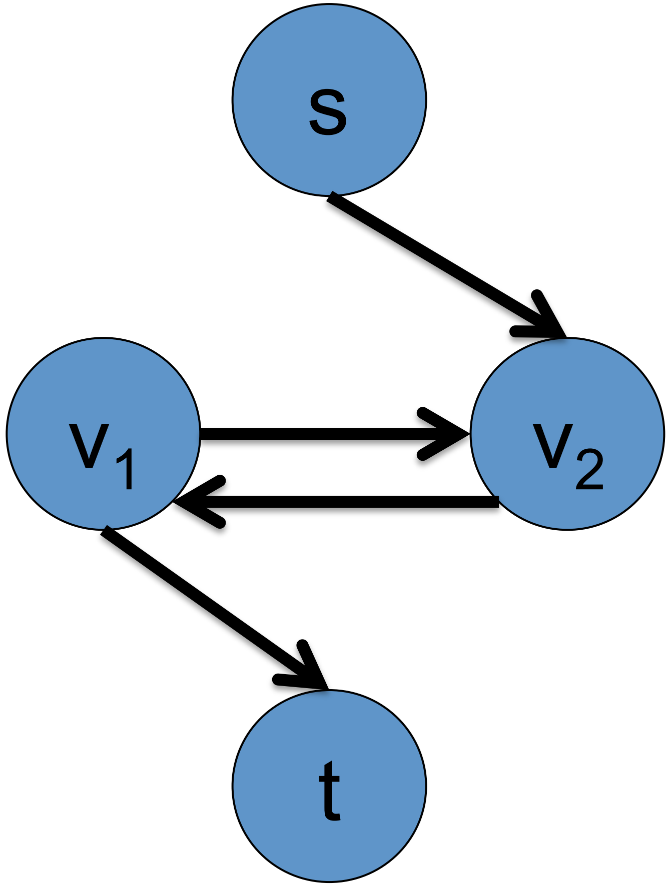



Update the residual graph.

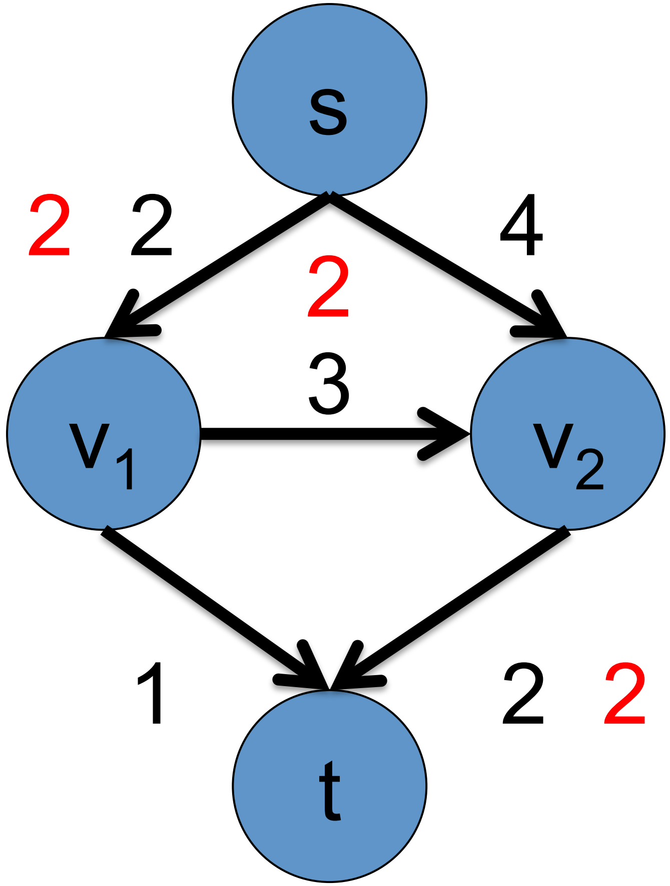
# Maximum Flow using Residual Graphs

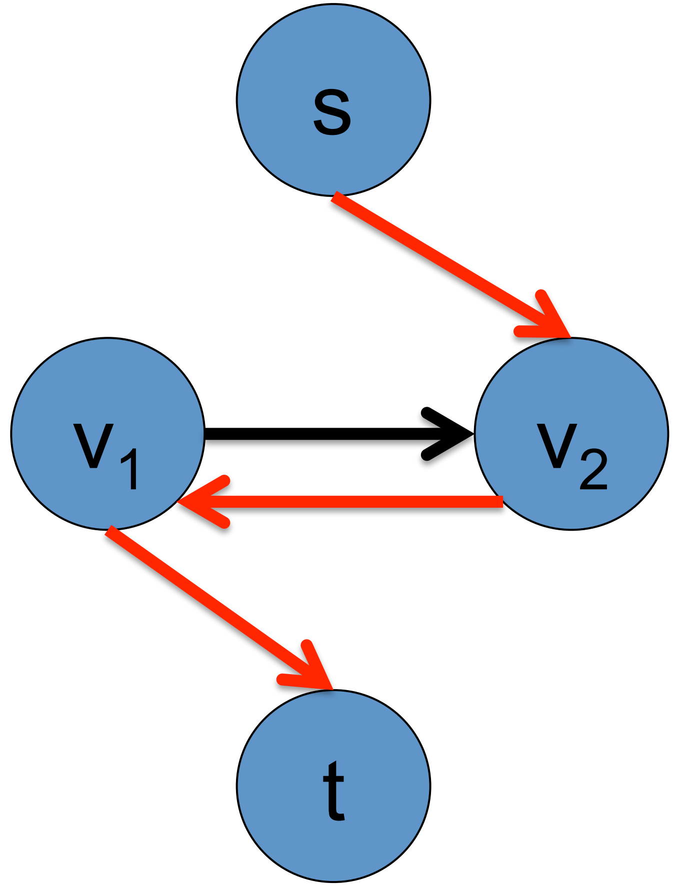



Find an s-t path in the residual graph.

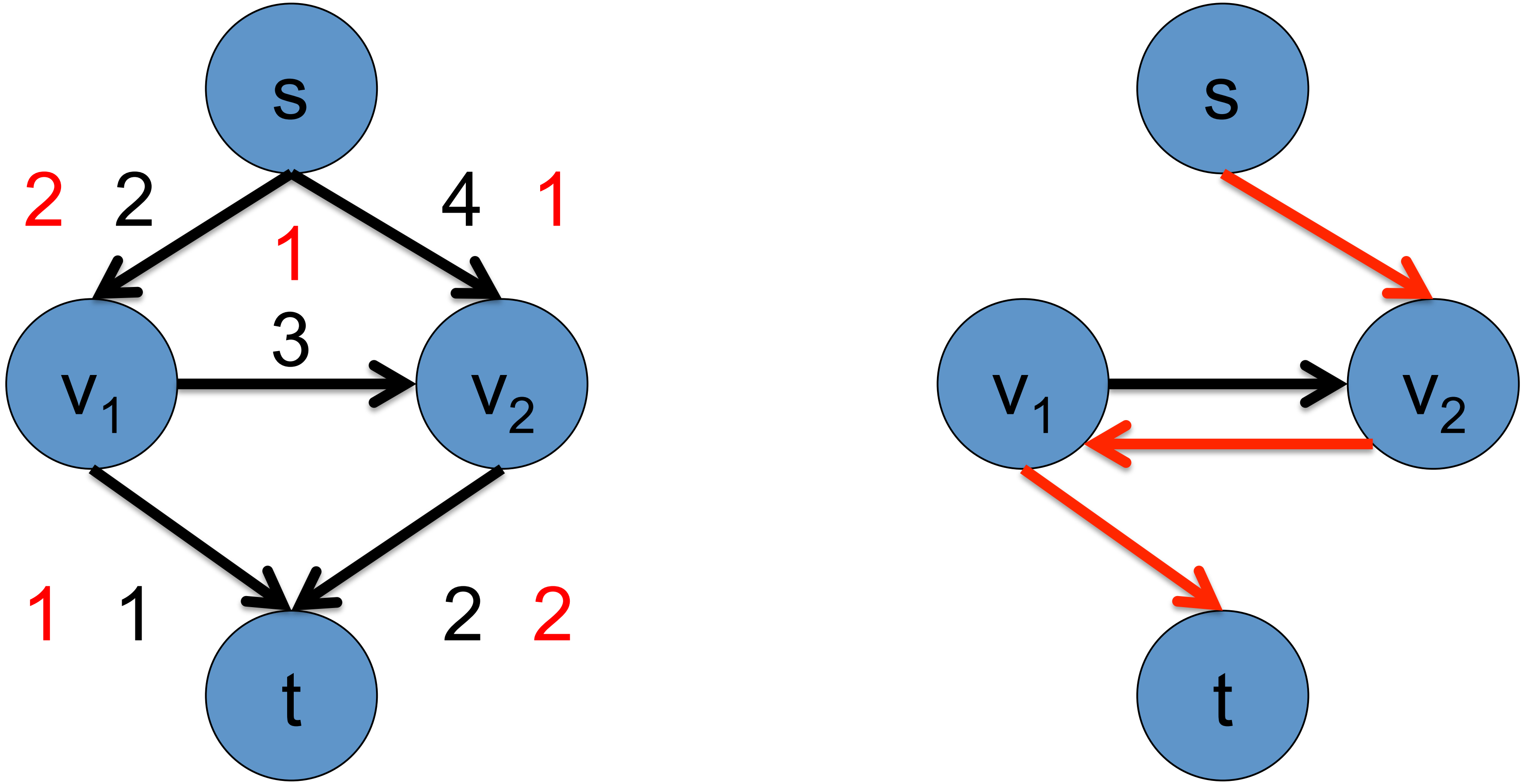
# Maximum Flow using Residual Graphs




Find an s-t path in the residual graph.

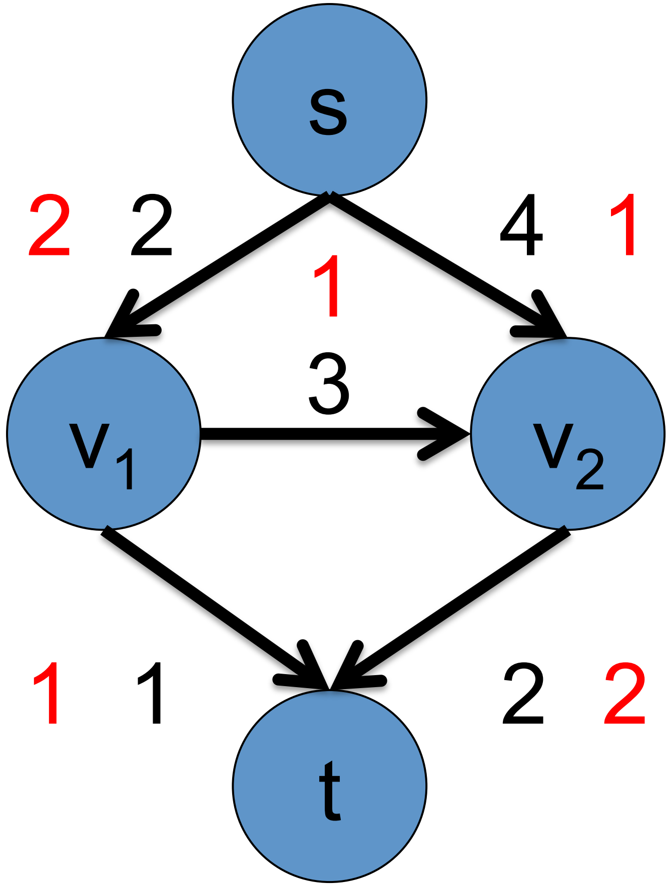
# Maximum Flow using Residual Graphs

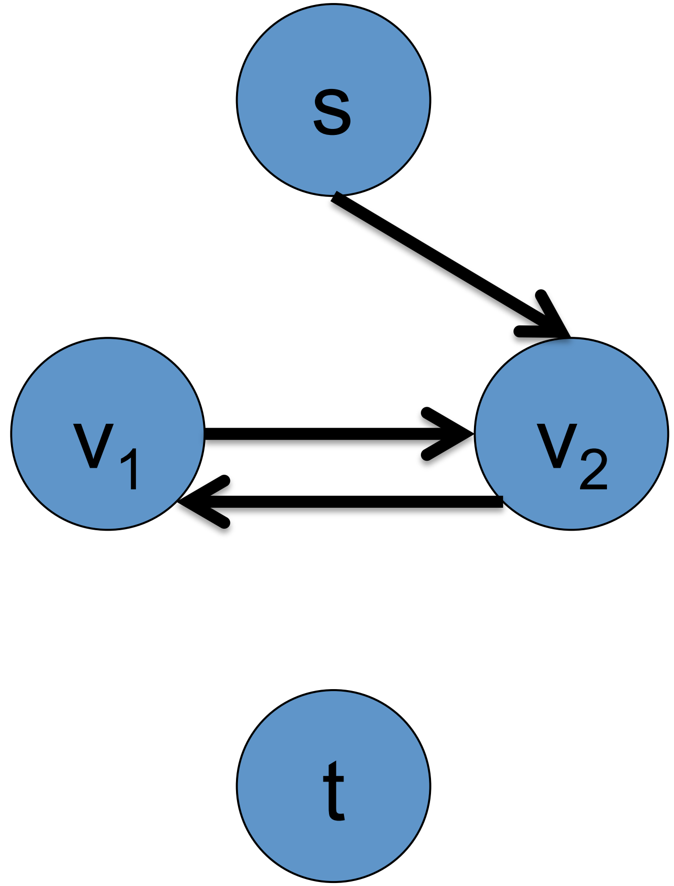



Choose maximum allowable value of K.

Add K to $(s,v_2)$ and $(v_1,t)$. Subtract K from $(v_1,v_2)$.

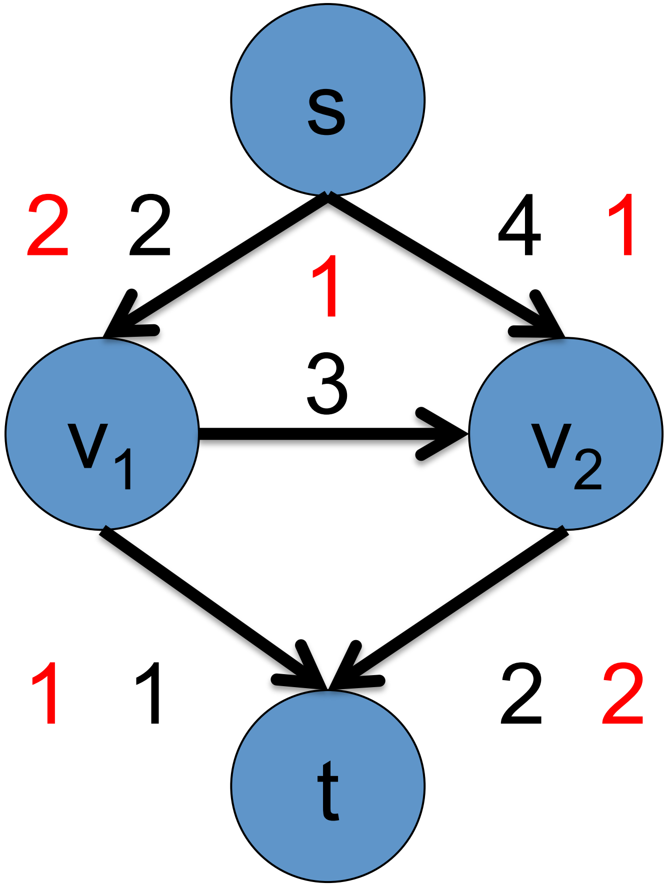
# Maximum Flow using Residual Graphs

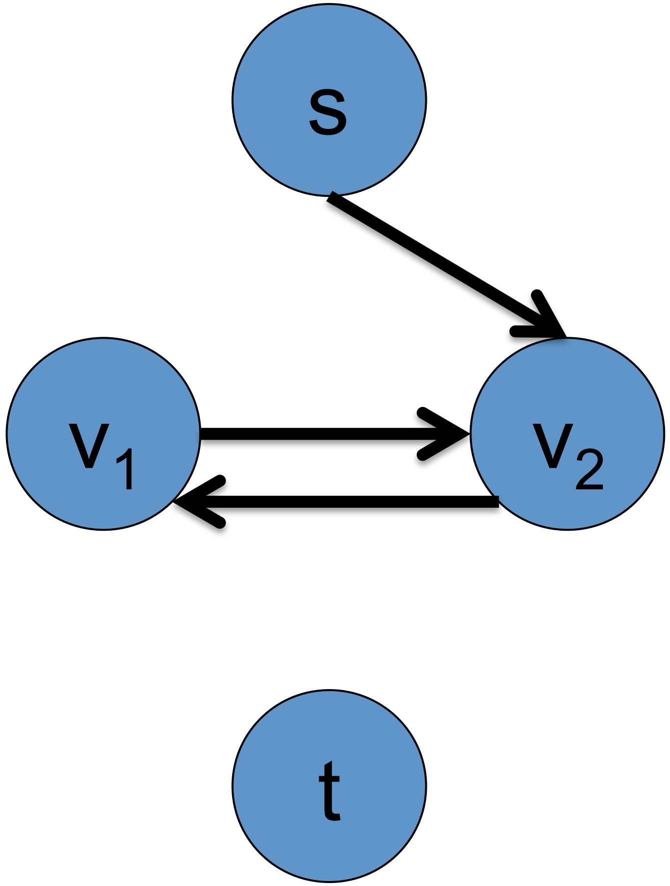



Update the residual graph.

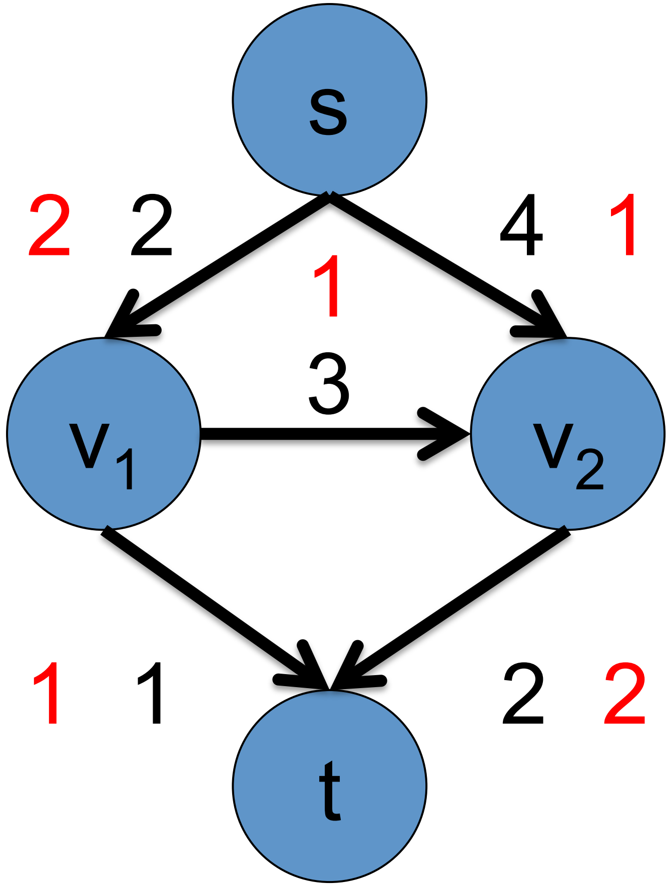
# Maximum Flow using Residual Graphs

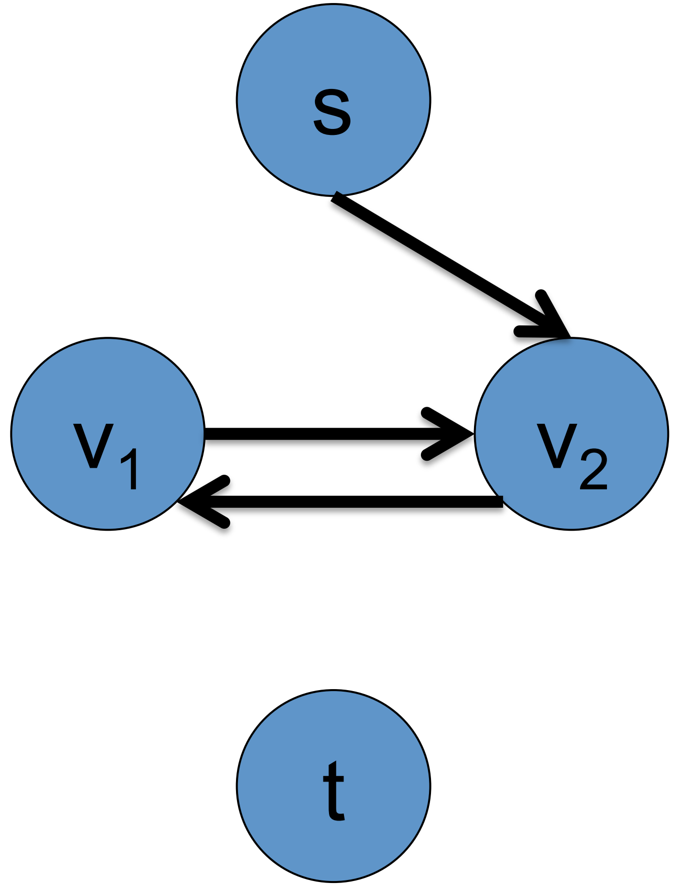



Find an s-t path in the residual graph.

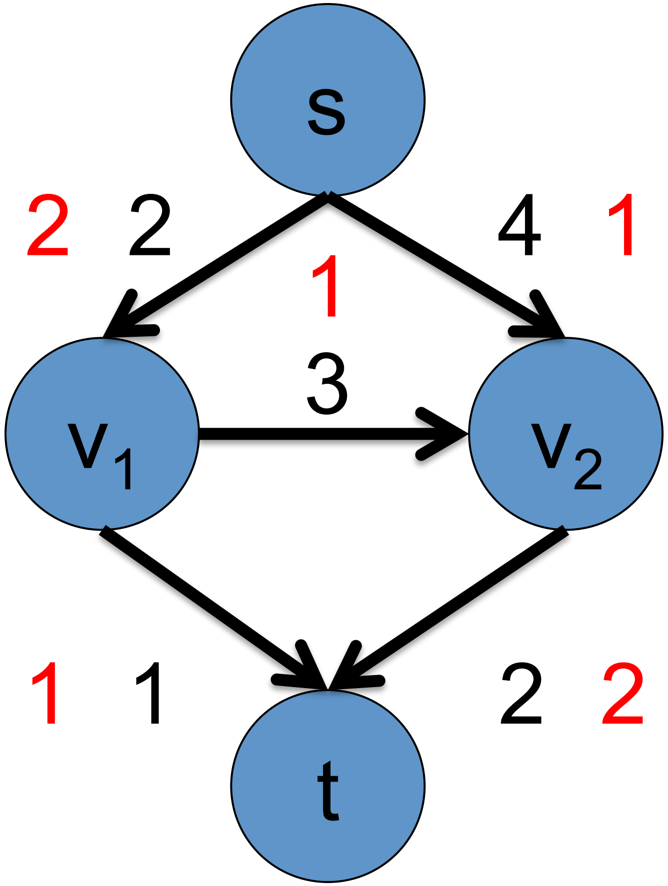
# Maximum Flow using Residual Graphs

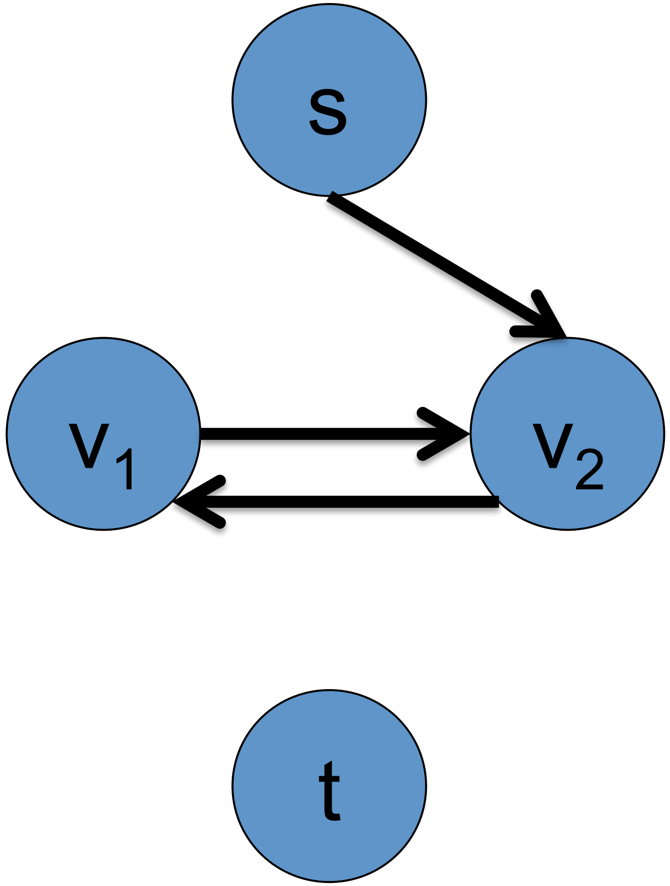



No more s-t paths. Stop.

# Maximum Flow using Residual Graphs




Correct Answer.