Generic Acceleration Schemes for Gradient-Based Optimization

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Collaborators

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Publications

H. Lin, J. Mairal and Z. Harchaoui. A Universal Catalyst for First-Order Optimization. Adv. NIPS 2015.

H. Lin, J. Mairal and Z. Harchaoui. QuickeNing: A Generic Quasi-Newton Algorithm for Faster Gradient-Based Optimization. 2017.

Focus of this work

Minimizing large finite sums

Consider the minimization of a large sum of convex functions

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

where each f_i is **smooth and convex** and ψ is a convex but not necessarily differentiable penalty, e.g., the ℓ_1-norm.

Goal of this work

- Design accelerated methods for minimizing **large finite sums**.
- Give a **generic acceleration schemes** which can be applied to previously un-accelerated algorithms.

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Two solutions: (2) QuickeNing (Quasi Newton);

Why do large finite sums matter?

Empirical risk minimization

$$\min_{x \in \mathbb{R}^p} \left\{ F(x) \triangleq \frac{1}{n} \sum_{i=1}^{n} f_i(x) + \psi(x) \right\},$$

- Typically, x represents **model parameters**.
- Each function f_i measures the **fidelity** of x to a data point.
- ψ is a **regularization function** to prevent overfitting.

For instance, given training data $(y_i, z_i)_{i=1,\dots,n}$ with features z_i in $\mathbb{R}^p$ and labels y_i in $\{-1, +1\}$, we may want to predict y_i by $\text{sign}(\langle z_i, x \rangle)$. The functions f_i measure how far the prediction is from the true label.

This would be a **classification problem with a linear model**.

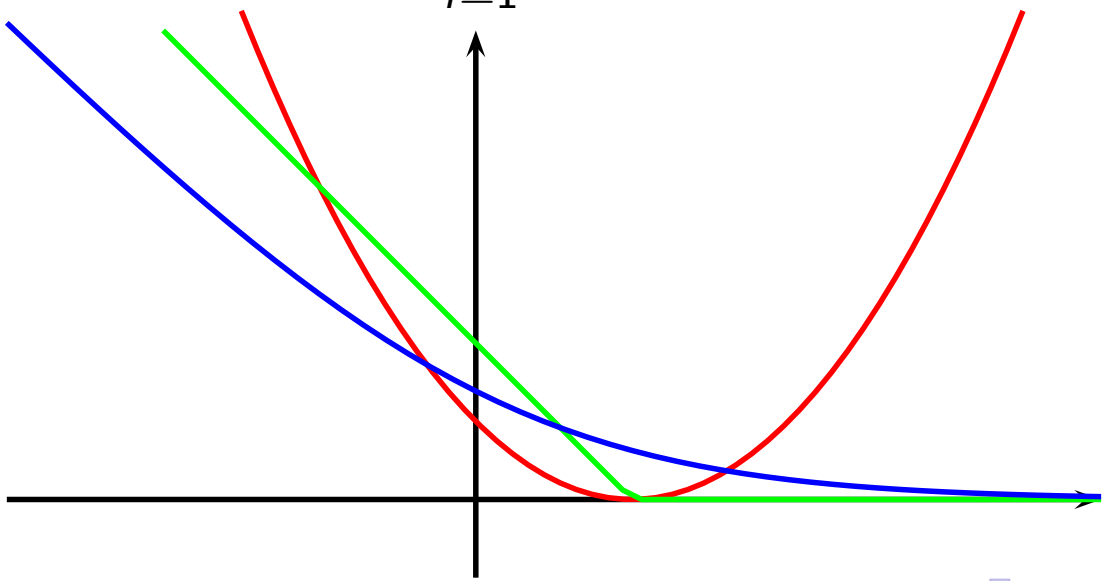
Why large finite sums matter?

A few examples

Ridge regression:
$$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}(y_i - \langle x, z_i\rangle)^2 + \frac{\lambda}{2}\|x\|_2^2.$$

Linear SVM:
$$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \max(0, 1 - y_i\langle x, z_i\rangle) + \frac{\lambda}{2}\|x\|_2^2.$$

Logistic regression:
$$\min_{x\in\mathbb{R}^p} \frac{1}{n}\sum_{i=1}^{n} \log\left(1 + e^{-y_i\langle x, z_i\rangle}\right) + \frac{\lambda}{2}\|x\|_2^2.$$



Why does the composite problem matter?

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The **squared ℓ_2-norm** penalizes large entries in x.

Why does the composite problem matter?

A few examples

Ridge regression: $$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2}(y_i - \langle x, z_i \rangle)^2 + \lambda \|x\|_1.$$

Linear SVM: $$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \max(0, 1 - y_i \langle x, z_i \rangle)^2 + \lambda \|x\|_1.$$

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When one knows in advance that x should be sparse, one should use a **sparsity-inducing** regularization such as the ℓ_1-norm.

[Chen et al., 1999, Tibshirani, 1996].

Gradient descent methods

Let us consider the composite problem

$$\min_{x \in \mathbb{R}^p} \{F(x) = f(x) + \psi(x)\},$$

where f is convex, differentiable with L-Lipschitz continuous gradient and ψ is convex, but not necessarily differentiable.

The classical forward-backward/ISTA algorithm

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \frac{1}{2} \left\| x - \left(x_{k-1} - \frac{1}{L} \nabla f(x_{k-1}) \right) \right\|_2^2 + \frac{1}{L} \psi(x).$$

- $F(x_k) - F^\star = O(1/k)$ for **convex** problems;
- $F(x_k) - F^\star = O((1 - \mu/L)^k)$ for μ-**strongly convex** problems;

[Nowak and Figueiredo, 2001, Daubechies et al., 2004, Combettes and Wajs, 2006, Beck and Teboulle, 2009, Wright et al., 2009, Nesterov, 2013]...

Accelerated gradient descent methods

Nesterov introduced in the 80's an acceleration scheme for the gradient descent algorithm. It was generalized later to the composite setting.

FISTA [Beck and Teboulle, 2009]

$$x_k \leftarrow \arg\min_{x \in \mathbb{R}^p} \frac{1}{2}\left\| x - \left(y_{k-1} - \frac{1}{L}\nabla f(y_{k-1}) \right) \right\|_2^2 + \frac{1}{L}\psi(x);$$

$$\text{Find } \alpha_k > 0 \text{ s.t. } \alpha_k^2 = (1-\alpha_k)\alpha_{k-1}^2 + \frac{\mu}{L}\alpha_k;$$

$$y_k \leftarrow x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$

- $F(x_k) - F^\star = O(1/k^2)$ for **convex** problems;
- $F(x_k) - F^\star = O((1-\sqrt{\mu/L})^k)$ for μ-**strongly convex** problems;
- Acceleration works in many practical cases.

see also [Nesterov, 1983, 2004, 2013]

What do we mean by "acceleration"?

Complexity analysis for large finite sums

Since f is a sum of n functions, computing ∇f requires computing n gradients ∇f_i. The complexity to reach an $\varepsilon-$solution is given below

	$\mu > 0$	$\mu = 0$
ISTA	$O\left(n\frac{L}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(\frac{nL}{\varepsilon}\right)$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(\frac{n\sqrt{L}}{\sqrt{\varepsilon}}\right)$

Remarks

- ε-solution means here $f(x_k) - f^\star \leq \varepsilon$.
- For $n = 1$, the rates of FISTA are optimal for a "first-order local black box" [Nesterov, 2004].
- For $n > 1$, the sum structure of f is not exploited.

Can we do better for large finite sums?

Several **randomized** algorithms are designed with one ∇f_i computed per iteration, which yields a better **expected computational complexity**.

	$\mu > 0$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$
SVRG, SAG, SAGA, SDCA, MISO, Finito	$O\left(\max\left(n, \frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$

SVRG, SAG, SAGA, SDCA, MISO, Finito improve upon FISTA when

$$\max\left(n, \frac{L}{\mu}\right) \leq n\sqrt{\frac{L}{\mu}} \quad \Leftrightarrow \sqrt{\frac{L}{\mu}} \leq n,$$

but they are not "accelerated" in the sense of Nesterov.

[Schmidt et al., 2013, Xiao and Zhang, 2014, Defazio et al., 2014a,b, Shalev-Shwartz and Zhang, 2012, Mairal, 2015, Zhang and Xiao, 2015]

Can we do even better for large finite sums?

Without vs with acceleration

	$\mu > 0$
FISTA	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$
SVRG, SAG, SAGA, SDCA, MISO, Finito	$O\left(\max\left(n, \frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$
Acc-SDCA	$\tilde{O}\left(\max\left(n, \sqrt{n\frac{L}{\mu}}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$

Acc-SDCA is due to Shalev-Shwartz and Zhang [2014].

- Acceleration occurs when $n \leq \frac{L}{\mu}$.
- see [Agarwal and Bottou, 2015] for discussions about optimality.

Challenge: can we accelerate these algorithms by a universal scheme for both convex and strongly convex objectives ?

Catalyst is coming

Main idea

Catalyst, a meta-algorithm

Given an algorithm $\mathcal{M}$ that can solve a convex problem "appropriately".

- At iteration k, rather than minimizing F, we use $\mathcal{M}$ to minimize a function G_k, defined as follows,

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

  up to accuracy ε_k, i.e., such that $G_k(x_k) - G_k^\star \leq \varepsilon_k$.
- Then compute the next prox-center y_k using an extrapolation step

$$y_k = x_k + \beta_k(x_k - x_{k-1}).$$

The choices of $\beta_k, \epsilon_k, \kappa$ are driven by the theoretical analysis.

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The choices of $\beta_k, \epsilon_k, \kappa$ are driven by the theoretical analysis.

Catalyst is a wrapper of $\mathcal{M}$ that yields an **accelerated** algorithm $\mathcal{A}$.

Sources of inspiration

In addition to accelerated proximal algorithms [Beck and Teboulle, 2009, Nesterov, 2013], several works have inspired Catalyst.

The inexact accelerated proximal point algorithm of Güler [1992].

- Catalyst is a variant of inexact accelerated PPA.
- Complexity analysis for **outer-loop only** with non practical inexactness criterion.

Accelerated SDCA of Shalev-Shwartz and Zhang [2014].

- Accelerated SDCA is an instance of inexact accelerated PPA.
- Complexity analysis **limited to μ-strongly convex objectives**.

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Other related work

[Frostig et al., 2015, Schmidt et al., 2011, Salzo and Villa, 2012, He and Yuan, 2012, Lan, 2015, Devolder et al., 2014].

This work

Contributions

- **Generic acceleration scheme**, which applies to previously unaccelerated algorithms such as SVRG, SAG, SAGA, SDCA, MISO, or Finito, and which is not taylored to finite sums.
- Provides explicit **support to non-strongly convex objectives**.
- Complexity analysis for μ-strongly convex objectives.
- Complexity analysis for non-strongly convex objectives.

Example of application

Garber and Hazan [2015] have used Catalyst to accelerate new principal component analysis algorithms based on convex optimization.

Appropriate $\mathcal{M}$ = Linear convergence rate when $\mu > 0$

Linear convergence rate

Consider a **strongly convex** minimization problem

$$\min_{z \in \mathbb{R}^p} H(z).$$

We say that an algorithm $\mathcal{M}$ has a **linear convergence rate** if $\mathcal{M}$ generates a sequence of iterates $(z_t)_{t \in \mathbb{N}}$ such that there exists $\tau_{\mathcal{M},H}$ in $(0,1)$ and a constant $C_{\mathcal{M},H}$ in $\mathbb{R}$ satisfying

$$H(z_t) - H^\star \leq C_{\mathcal{M},H}(1 - \tau_{\mathcal{M},H})^t. \tag{1}$$

- $\tau_{\mathcal{M},H}$ depends usually on the **condition number** L/μ, e.g., $\tau_{\mathcal{M},H} = \mu/L$ for ISTA and $\tau_{\mathcal{M},H} = \sqrt{\mu/L}$ for FISTA.
- $C_{\mathcal{M},H}$ depends usually on $H(z_0) - H^*$.

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Important message: we do not make any assumption for non strongly convex objectives. It is possible that $\mathcal{M}$ is not even defined for $\mu = 0$.

Catalyst action

Catalyst action

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

- G_k is always strongly convex as long as F is convex.
- When F is strongly convex, the condition number of G_k is better than that of F since $\frac{L+\kappa}{\mu+\kappa} < \frac{L}{\mu}$.

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Minimizing G_k is easier than minimizing F !

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Minimizing G_k is easier than minimizing F !

- If $\kappa \gg 1$, then minimizing G_k is easy;
- If $\kappa \approx 0$, then G_k is a good approximation of F.

We will choose κ to optimize the computational complexity.

Convergence analysis

An analysis in two stages

$$G_k(x) \triangleq F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2,$$

x_k is a approximate minimizer of G_k such that $G_k(x_k) - G_k^* \leq \epsilon_k$.

- Outer loop: once we obtain the sequence $(x_k)_{k\in\mathbb{N}}$, what can we say about the convergence rate of $F(x_k) - F^*$?
  $\Rightarrow$ Wisely choose (y_k) and control the accumulation of errors.
- Inner loop: how much effort do we need to obtain a x_k with accuracy ϵ_k?
  $\Rightarrow$ Wisely choose the starting point.

Choice of $(y_k)_{k\in\mathbb{N}}$

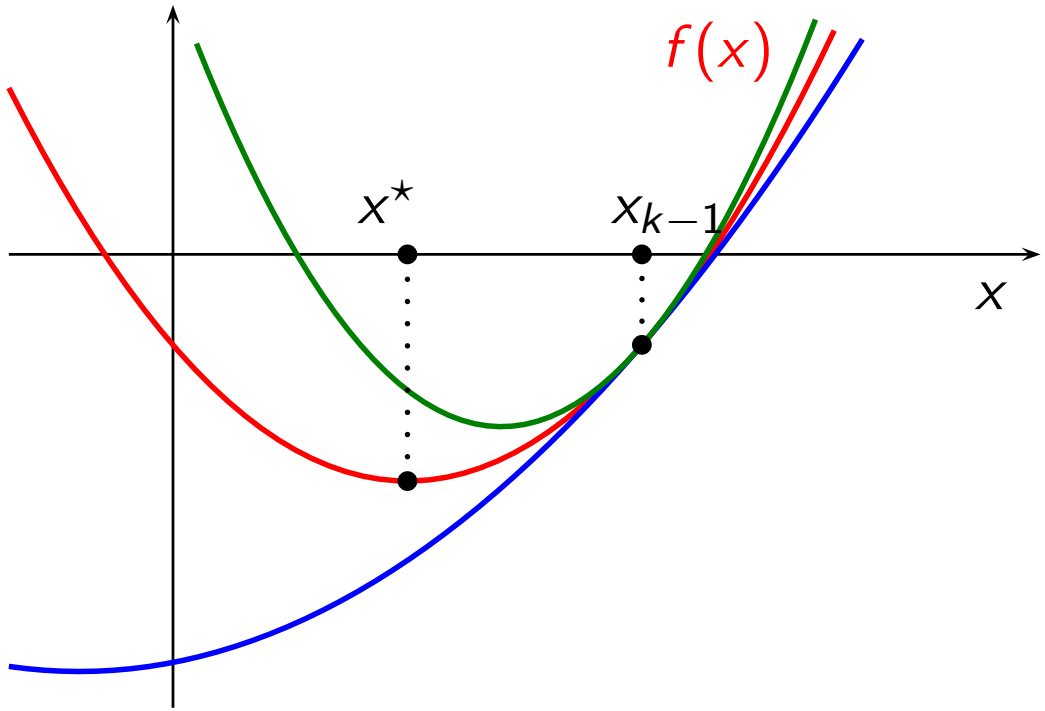
Extrapolation

$$y_k = x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$

- This update is identical to Nesterov's accelerated gradient descent or FISTA.
- Unfortunately, the literature does not provide any simple geometric explanation why it yields an acceleration...
- The construction is purely theoretical by using a mechanism introduced by Nesterov, called **"estimate sequence"**.

How does "acceleration" work?

If f is μ-strongly convex and ∇f is L-Lipschitz continuous



- $f(x) \leq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{L}{2}\|x - x_{k-1}\|_2^2$;
- $f(x) \geq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{\mu}{2}\|x - x_{k-1}\|_2^2$;

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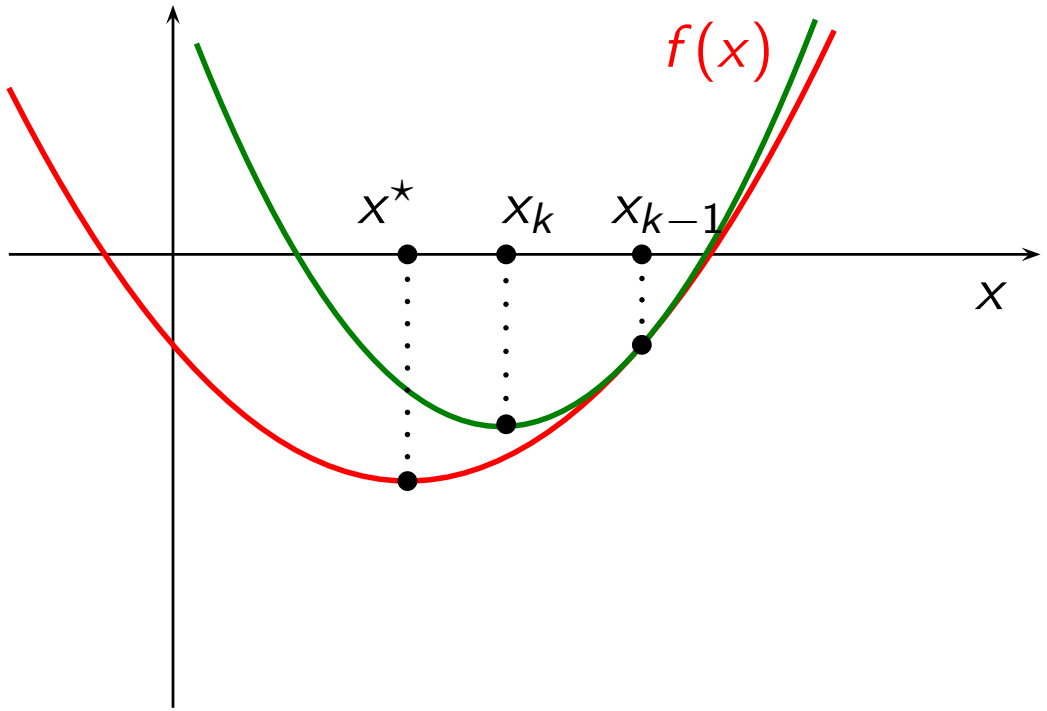

- $f(x) \leq f(x_{k-1}) + \nabla f(x_{k-1})^\top (x - x_{k-1}) + \frac{L}{2}\|x - x_{k-1}\|_2^2$;
- $\boxed{x_k = x_{k-1} - \frac{1}{L}\nabla f(x_{k-1}) \text{ (gradient descent step).}}$

How does "acceleration" work?

Definition of estimate sequence [Nesterov].

A pair of sequences $(\varphi_k)_{k\geq 0}$ and $(\lambda_k)_{k\geq 0}$, with $\varphi_k : \mathbb{R}^p \to \mathbb{R}$ and $\lambda_k \geq 0$, is called an **estimate sequence** of function F if

- $\lambda_k \to 0$;
- $\varphi_k(x) \leq (1-\lambda_k)F(x) + \lambda_k \varphi_0(x)$, for any k, x;
- There exists a sequence $(x_k)_{k\geq 0}$ such that

$$F(x_k) \leq \varphi_k^\star \stackrel{\triangle}{=} \min_{x\in\mathbb{R}^p} \varphi_k(x).$$

Remarks

- φ_k is neither an upper-bound, nor a lower-bound;
- Finding the right estimate sequence is often nontrivial.

Convergence of outer-loop algorithm

Analysis for μ-strongly convex objective functions

Choose $\alpha_0 = \sqrt{q}$ with $q = \mu/(\mu + \kappa)$ and

$$\epsilon_k = \frac{2}{9}(F(x_0) - F^*)(1-\rho)^k \quad \text{with} \quad \rho < \sqrt{q}.$$

Then, the algorithm generates iterates $(x_k)_{k\geq 0}$ such that

$$F(x_k) - F^* \leq C(1-\rho)^{k+1}(F(x_0) - F^*) \quad \text{with} \quad C = \frac{8}{(\sqrt{q}-\rho)^2}.$$

In practice

- Choice of ρ can safely be set to $\rho = 0.9\sqrt{q}$.
- Choice of $(\varepsilon_k)_{k\geq 0}$ typically follows from a duality gap at x_0. When F is non-negative, we can set $\varepsilon_k = (2/9)F(x_0)(1-\rho)^k$.

Convergence of outer-loop algorithm

Analysis for non-strongly convex objective functions, $\mu = 0$

Choose $\alpha_0 = (\sqrt{5}-1)/2$ and

$$\epsilon_k = \frac{2(F(x_0)-F^*)}{9(k+2)^{4+\eta}} \quad \text{with } \eta > 0.$$

Then, the meta-algorithm generates iterates $(x_k)_{k\geq 0}$ such that

$$F(x_k) - F^* \leq \frac{8}{(k+2)^2}\left(\left(1+\frac{2}{\eta}\right)^2 (F(x_0)-F^*) + \frac{\kappa}{2}\|x_0 - x^*\|^2\right). \tag{2}$$

In practice

- Choice of η can be set to $\eta = 0.1$.

How many iterates of $\mathcal{M}$ do we need to obtain x_k?

Control of inner-loop complexity

For minimizing G_k, consider a method $\mathcal{M}$ generating iterates $(z_t)_{t\geq 0}$ with linear convergence rate

$$G_k(z_t) - G_k^\star \leq A(1-\tau_{\mathcal{M}})^t(G_k(z_0) - G_k^\star).$$

Then by choosing $z_0 = x_{k-1}$, the precision ε_k is reached with at most

- A constant number of iterations $T_{\mathcal{M}}$ when $\mu > 0$;
- A logarithmic increasing number of iterations $T_{\mathcal{M}} \log(k+2)$ when $\mu = 0$.

where $T_{\mathcal{M}} = \tilde{O}(1/\tau_{\mathcal{M}})$ is independent of k.

Global computational complexity

Analysis for μ-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^\star \leq C\left(1 - \frac{\rho}{T_{\mathcal{M}}}\right)^s (F(x_0) - F^*). \tag{3}$$

where F_s is the objective function value obtained after performing $s = kT_{\mathcal{M}}$ iterations of the method $\mathcal{M}$. As a result,

$$\tau_{\mathcal{A},F} = \frac{\rho}{T_{\mathcal{M}}} = \tilde{O}(\tau_{\mathcal{M}}\sqrt{\mu}/\sqrt{\mu+\kappa}),$$

where $\tau_{\mathcal{M}}$ typically depends on κ (the greater, the faster).

κ will be chosen to maximize the ratio $\tau_{\mathcal{M}}/\sqrt{\mu+\kappa}$.

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where $\tau_{\mathcal{M}}$ typically depends on κ (the greater, the faster).

e.g., $\kappa = L - 2\mu$ when $\tau_{\mathcal{M}} = \frac{\mu+\kappa}{L+\kappa} \Rightarrow \tau_{\mathcal{A}} = \tilde{O}\left(\sqrt{\frac{\mu}{L}}\right)$.

Global computational complexity

Analysis for non-strongly convex objective functions

The global convergence rate of the accelerated algorithm $\mathcal{A}$ is

$$F_s - F^* \leq \frac{8 T_{\mathcal{M}}^2 \log^2(s)}{s^2} \left(\left(1 + \frac{2}{\eta}\right)^2 (F(x_0) - F^*) + \frac{\kappa}{2} \|x_0 - x^*\|^2 \right) .$$

If $\mathcal{M}$ is a first-order method, this rate is **near-optimal**, up to a logarithmic factor, when compared to the optimal rate $O(1/s^2)$, which may be the price to pay for using a generic acceleration scheme.

κ will be chosen to maximize the ratio $\tau_{\mathcal{M}}/\sqrt{L+\kappa}$

Applications

Expected computational complexity in the regime $n \leq L/\mu$ when $\mu > 0$,

	$\mu > 0$	$\mu = 0$	Catalyst $\mu > 0$	Cat. $\mu = 0$
FG	$O\left(n\left(\frac{L}{\mu}\right)\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(n\frac{L}{\varepsilon}\right)$	$\tilde{O}\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$\tilde{O}\left(n\frac{\sqrt{L}}{\sqrt{\varepsilon}}\right)$
SAG	$O\left(\frac{L}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$		$\tilde{O}\left(\sqrt{\frac{nL}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	
SAGA				
Finito/MISO		NA		
SDCA				
SVRG	$O\left(\frac{L'}{\mu}\log\left(\frac{1}{\varepsilon}\right)\right)$		$\tilde{O}\left(\sqrt{\frac{nL'}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	
Acc-FG	$O\left(n\sqrt{\frac{L}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	$O\left(n\frac{\sqrt{L}}{\sqrt{\varepsilon}}\right)$	no acceleration	
Acc-SDCA	$\tilde{O}\left(\sqrt{\frac{nL}{\mu}}\log\left(\frac{1}{\varepsilon}\right)\right)$	NA		

A fresh look at Catalyst

A fresh look at Catalyst

The Moreau-Yosida regularization of a convex function f is defined as

$$F(x) = \min_{z \in \mathbb{R}^p} \left\{ f(x) + \frac{\kappa}{2}\|x - z\|^2 \right\},$$

and call $p(x)$ the unique solution of the problem.

The equivalence property

F is convex and minimizing f and F are equivalent in the sense that

$$\min_{x \in \mathbb{R}^p} F(x) = \min_{x \in \mathbb{R}^p} f(x).$$

The **minimizers of f and F coincide with each other**.

A fresh look at Catalyst

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The smoothness properties

- F is **continuously differentiable** even when f is not and

$$\nabla F(x) = \kappa(x - p(x)),$$

  The gradient ∇F is Lipschitz continuous with constant $L_F = \kappa$.
- When f is μ-strongly convex, F is μ_F-strongly convex with constant $\mu_F = \frac{\mu\kappa}{\mu+\kappa}$.
- $\Rightarrow$ When $\mu > 0$, the condition number of F is $1 + \frac{\kappa}{\mu}$.

A fresh look at Catalyst

A naive approach consists of minimizing F instead of f with a method designed for smooth optimization. Consider indeed

$$x_{k+1} = x_k - \frac{1}{\kappa}\nabla F(x_k).$$

By rewriting the gradient $\nabla F(x_k)$ as $\kappa(x_k - p(x_k))$, we obtain

$$x_{k+1} = p(x_k) = \arg\min_{z \in \mathbb{R}^p} \left\{ f(z) + \frac{\kappa}{2}\|z - x_k\|^2 \right\}.$$

This is exactly the **proximal point algorithm**.

A fresh look at Catalyst

Consider now

$$x_{k+1} = y_k - \frac{1}{\kappa}\nabla F(y_k) \quad \text{and} \quad y_{k+1} = x_{k+1} + \beta_{k+1}(x_{k+1} - x_k),$$

where β_{k+1} is a Nesterov-like extrapolation parameter. We may now rewrite the update using the value of ∇F, which gives:

$$x_{k+1} = p(y_k) \quad \text{and} \quad y_{k+1} = x_{k+1} + \beta_{k+1}(x_{k+1} - x_k)$$

This is the **accelerated proximal point algorithm** of Güler [1992].

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What is the advantage of these approaches?

F may be better conditioned than f when $1 + \kappa/\mu \leq L/\mu$;

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What is the advantage of these approaches?

F may be better conditioned than f when $1 + \kappa/\mu \leq L/\mu$;

But...

Computing $p(y_k)$ has a cost!

A fresh look at Catalyst

Catalyst is a particular accelerated proximal point algorithm with **inexact gradients** [Güler, 1992].

$$x_{k+1} \approx p(y_k) \quad \text{and} \quad y_{k+1} = x_{k+1} + \beta_{k+1}(x_{k+1} - x_k)$$

The quantity x_{k+1} is obtained by approximately solving using an optimization method $\mathcal{M}$:

$$x_{k+1} \approx \arg\min_{x \in \mathbb{R}^p} \left\{ h_k(x) \triangleq f(x) + \frac{\kappa}{2}\|x - y_k\|^2 \right\},$$

such that $h_k(x_{k+1}) - h_k^* \leq \epsilon_k$.

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Catalyst provides Nesterov's acceleration to $\mathcal{M}$ with...

- the right κ, sequence $(\varepsilon_k)_{k \geq 0}$, and restart strategy for $\mathcal{M}$.
- global complexity analysis resulting in theoretical acceleration.

QuickeNing

QuickeNing is a **limited memory Quasi-Newton algorithm with inexact gradients** applied to the smoothed function F.

Main features

- uses an optimization method $\mathcal{M}$ to solve the sub-problems.
- if $\mathcal{M}$ is compatible with prox, so is QuickeNing.
- **linear convergence rate for strongly-convex functions**.
- **no need for a line-search and easy initialization** of B_0.

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Theory vs practice

- global theoretical complexity is not as good as Catalyst.
- in practice, outperforms Catalyst for ill-conditioned problems.

QuickeNing

Related work

- L-BFGS with inexact gradients [Friedlander and Schmidt, 2012].
- Quasi-Newton on Moreau-Yosida regularization [Burke and Qian, 2000, Chen and Fukushima, 1999, Fuentes et al., 2012, Fukushima and Qi, 1996].

Our contributions

- **practical** inexactness criterion and dedicated L-BFGS rule with no line search.
- **global complexity** with both inner- and outer-loop analysis.
- parameter choices that ensure **linear convergence rate for strongly-convex problems**.

Experiments about Catalyst

Experiments with MISO/SAG/SAGA

ℓ_2-logistic regression formulation

Given some data (y_i, z_i), with y_i in $\{-1, +1\}$ and z_i in $\mathbb{R}^p$, minimize

$$\min_{x \in \mathbb{R}^p} \frac{1}{n} \sum_{i=1}^{n} \log(1 + e^{-y_i x^\top z_i}) + \frac{\mu}{2} \|x\|_2^2,$$

μ is the regularization parameter and the strong convexity modulus.

Datasets

name	rcv1	real-sim	covtype	ocr	alpha
n	781 265	72 309	581 012	2 500 000	250 000
p	47 152	20 958	54	1 155	500

Experiments with MISO/SAG/SAGA

The complexity analysis is not just a theoretical exercise since it provides the values of $\kappa, \varepsilon_k, \beta_k$, which are required in concrete implementations.

Here, **theoretical values match practical ones**.

Restarting

The theory tells us to restart $\mathcal{M}$ with x_{k-1}. For SDCA/Finito/MISO, the theory tells us to use instead $x_{k-1} + \frac{\kappa}{\mu+\kappa}(y_{k-1} - y_{k-2})$. We also tried this as a heuristic for SAG and SAGA.

One-pass heuristic

constrain $\mathcal{M}$ to always perform at most n iterations in inner loop; we call this variant AMISO2 for MISO, whereas AMISO1 refers to the regular "vanilla" accelerated variant; idem to accelerate SAG and SAGA.

Experiments without strong convexity, $\mu = 0$

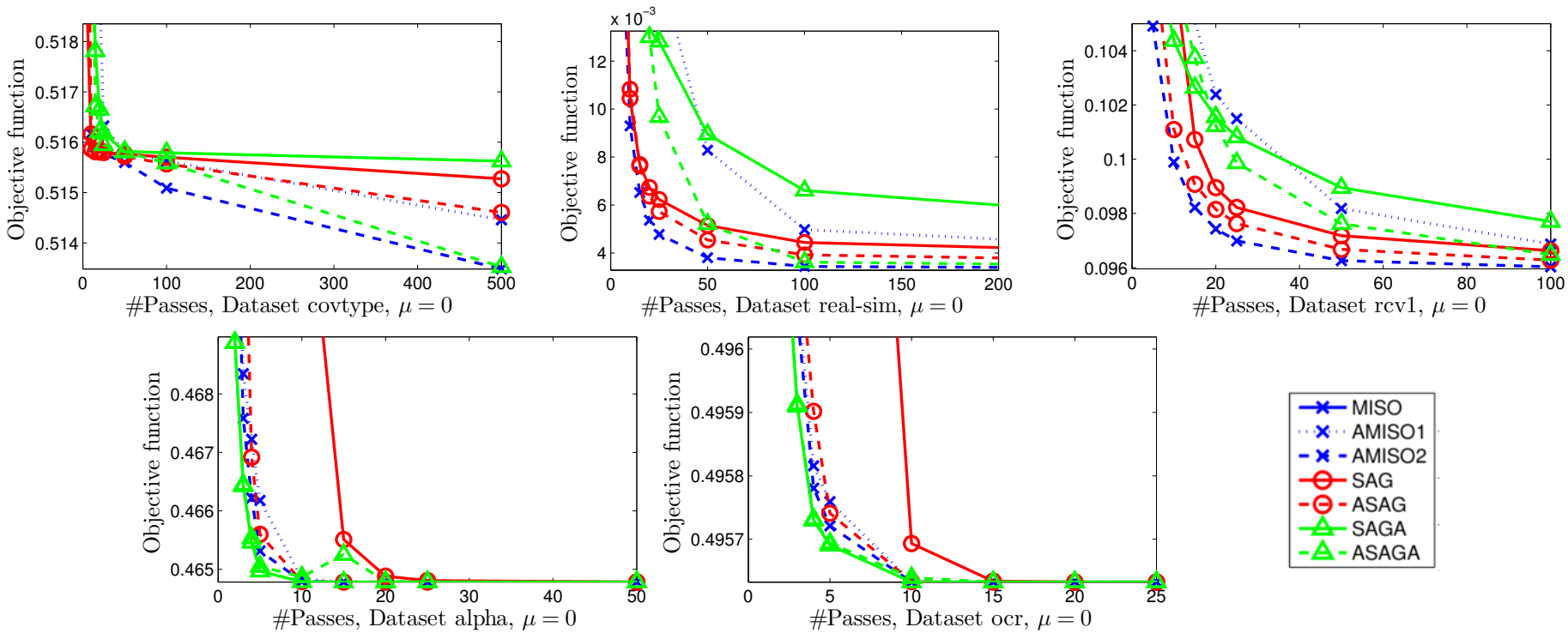

Figure: Objective function value for different number of passes on data.

Conclusions

- SAG, SAGA are accelerated when they do not perform well already;
- AMISO2 $\geq$ AMISO1 (vanilla), MISO does not apply.

Experiments without strong convexity, $\mu = 10^{-1}/n$

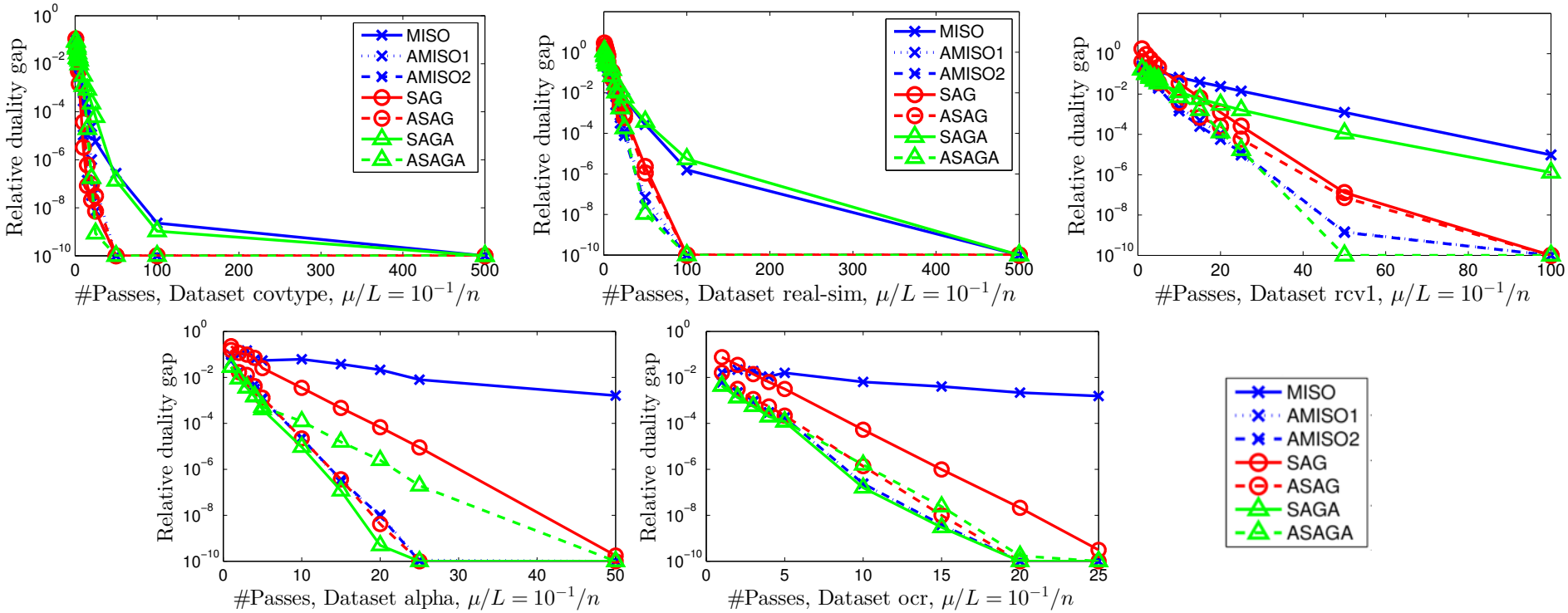

Figure: Relative duality gap (log-scale) for different number of passes on data.

Conclusions

- SAG, SAGA are not always accelerated, but often.
- AMISO2,AMISO1 $\gg$ MISO.

Experiments without strong convexity, $\mu = 10^{-3}/n$

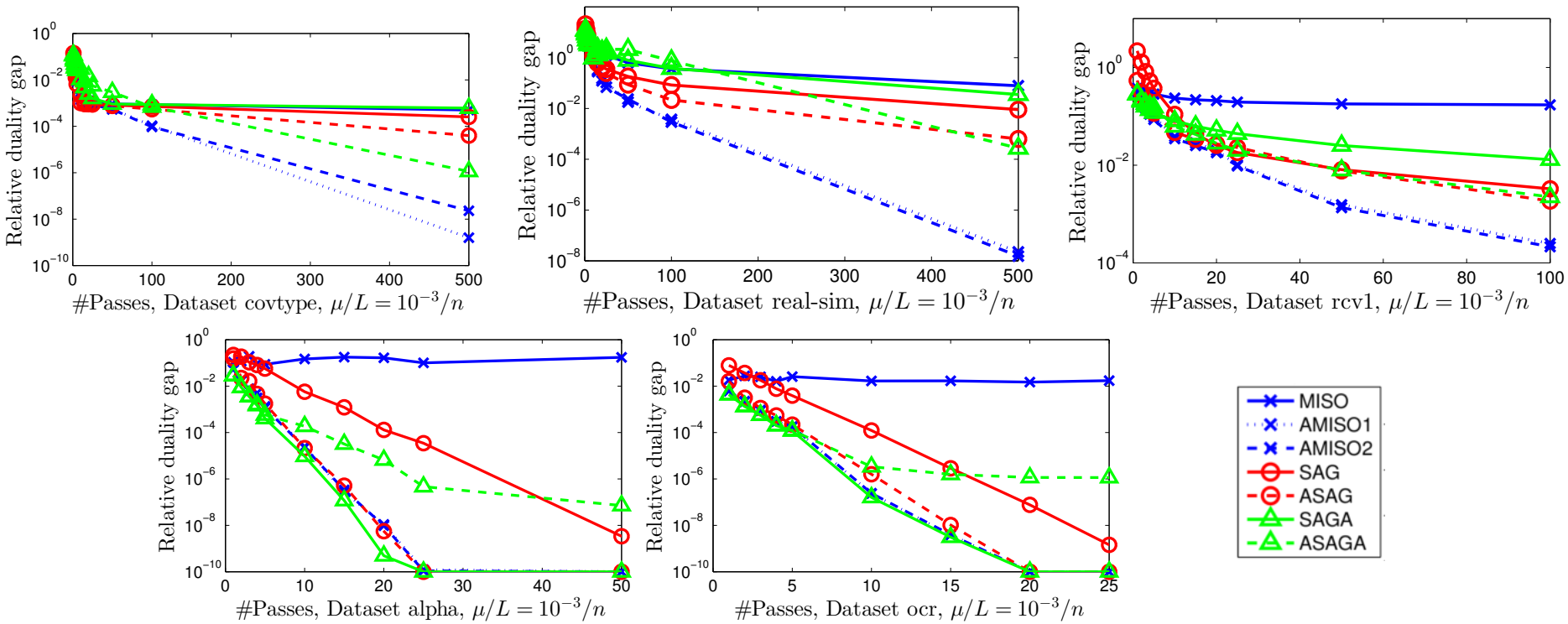

Figure: Relative duality gap (log-scale) for different number of passes on data.

Conclusions

- same conclusions as $\mu = 10^{-1}/n$;
- μ is so small that (unaccelerated) MISO becomes numerically unstable.

General conclusions about Catalyst

Summary: lots of nice features

- Simple acceleration scheme with broad application range.
- Recover near-optimal rates for known algorithms.
- Effortless implementation.

... but also lots of unsolved problems

- Acceleration occurs when $n \leq L/\mu$; otherwise, the "unaccelerated" complexity $O(n \log(1/\varepsilon))$ seems unbeatable.
- μ is an estimate of the true strong convexity parameter $\mu' \geq \mu$.
- μ is the global strong convexity parameter, not a local one $\mu^\star \geq \mu$.
- When $n \leq L/\mu$, but $n \geq L/(\mu' \text{ or } \mu^\star)$, a method $\mathcal{M}$ that adapts to the unknown strong convexity may be impossible to accelerate.
- The optimal restart for $\mathcal{M}$ is not yet fully understood.

Thank you for your attention!

Catalyst, the algorithm

Algorithm 1 Catalyst

input initial estimate $x_0 \in \mathbb{R}^p$, parameters κ and α_0, sequence $(\varepsilon_k)_{k\geq 0}$, optimization method $\mathcal{M}$; initialize $q = \mu/(\mu+\kappa)$ and $y_0 = x_0$;

1: **while** the desired stopping criterion is not satisfied **do**

2: Find an approx. solution x_k using $\mathcal{M}$ s.t. $G_k(x_k) - G_k^\star \leq \varepsilon_k$

$$x_k \approx \operatorname*{arg\,min}_{x\in\mathbb{R}^p} \left\{ G_t(x) \stackrel{\triangle}{=} F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|^2 \right\}$$

3: Compute $\alpha_k \in (0,1)$ from equation $\alpha_k^2 = (1-\alpha_k)\alpha_{k-1}^2 + q\alpha_k$;

4: Compute

$$y_k = x_k + \beta_k(x_k - x_{k-1}) \quad \text{with} \quad \beta_k = \frac{\alpha_{k-1}(1-\alpha_{k-1})}{\alpha_{k-1}^2 + \alpha_k}.$$

5: **end while**

output x_k (final estimate).

Ideas of the proofs

Main theorem

Let us denote

$$\lambda_k = \prod_{i=0}^{k-1}(1-\alpha_i), \tag{4}$$

where the α_i's are defined in Catalyst. Then, the sequence $(x_k)_{k\geq 0}$ satisfies

$$F(x_k) - F^* \leq \lambda_k \left(\sqrt{S_k} + 2\sum_{i=1}^{k} \sqrt{\frac{\epsilon_i}{\lambda_i}} \right)^2, \tag{5}$$

where $F^\star$ is the minimum value of F and

$$S_k = F(x_0) - F^* + \frac{\gamma_0}{2}\|x_0 - x^*\|^2 + \sum_{i=1}^{k} \frac{\epsilon_i}{\lambda_i} \quad \text{where} \quad \gamma_0 = \frac{\alpha_0\left((\kappa+\mu)\alpha_0 - \mu\right)}{1-\alpha_0}, \tag{6}$$

where $x^\star$ is a minimizer of F.

Ideas of the proofs

Then, the theorem will be used with the following lemma to control the convergence rate of the sequence $(\lambda_k)_{k\geq 0}$, whose definition follows the classical use of estimate sequences. This will provide us convergence rates both for the strongly convex and non-strongly convex cases.

Lemma 2.2.4 from Nesterov [2004]

If in the quantity γ_0 defined in (6) satisfies $\gamma_0 \geq \mu$, then the sequence $(\lambda_k)_{k\geq 0}$ from (4) satisfies

$$\lambda_k \leq \min\left\{(1-\sqrt{q})^k, \frac{4}{\left(2+k\sqrt{\frac{\gamma_0}{\kappa+\mu}}\right)^2}\right\}, \tag{7}$$

where $q \stackrel{\triangle}{=} \mu/(\mu+\kappa)$.

Ideas of proofs

Step 1: build an approximate estimate sequence

- Remember that in general, we build φ_k from φ_{k-1} as follows

$$\varphi_k(x) \stackrel{\triangle}{=} (1-\alpha_k)\varphi_{k-1}(x) + \alpha_k d_k(x),$$

where d_k is a lower bound.

- Here, a natural lower bound would be

$$F(x) \geq d_k(x) \stackrel{\triangle}{=} F(x_k^*) + \langle \kappa(y_{k-1} - x_k^*), x - x_k^* \rangle + \frac{\mu}{2}\|x - x_k^*\|^2,$$

where $x_k^\star \stackrel{\triangle}{=} \arg\min_{x \in \mathbb{R}^p} \left\{ G_k(x) \stackrel{\triangle}{=} F(x) + \frac{\kappa}{2}\|x - y_{k-1}\|_2^2 \right\}$.

- But $x_k^\star$ is unknown! Then, use instead $d_k'(x)$ defined as

$$d_k'(x) \stackrel{\triangle}{=} F(x_k) + \langle \kappa(y_{k-1} - x_k), x - x_k \rangle + \frac{\mu}{2}\|x - x_k\|^2.$$

Ideas of proofs

Step 2: Relax the condition $F(x_k) \leq \varphi_k^\star$.

- We can show that Catalyst generates iterates $(x_k)_{k\geq 0}$ such that

$$F(x_k) \leq \phi_k^* + \xi_k,$$

where the sequence $(\xi_k)_{k\geq 0}$ is defined by $\xi_0 = 0$ and

$$\xi_k = (1-\alpha_{k-1})(\xi_{k-1} + \varepsilon_k - (\kappa+\mu)\langle x_k - x_k^*, x_{k-1} - x_k\rangle).$$

- The sequences $(\alpha_k)_{k\geq 0}$ and $(y_k)_{k\geq 0}$ are chosen in such a way that all the terms involving $y_{k-1} - x_k$ are cancelled.
- We will control later the quantity $x_k - x_k^*$ by strong convexity of G_k:

$$\frac{\kappa+\mu}{2}\|x_k - x_k^*\|_2^2 \leq G_k(x_k) - G_k^\star \leq \varepsilon_k.$$

Ideas of proofs

Step 3: Control how this errors sum up together.

- Do cumbersome calculus.

Catalyst in practice

General strategy and application to randomized algorithms

Calculating the iteration-complexity decomposes into three steps:

1. When F is μ-strongly convex, find κ that maximizes the ratio $\tau_{\mathcal{M},G_k}/\sqrt{\mu+\kappa}$ for algorithm $\mathcal{M}$. When F is non-strongly convex, maximize instead the ratio $\tau_{\mathcal{M},G_k}/\sqrt{L+\kappa}$.
2. Compute the upper-bound of the number of outer iterations k_{out} using the theorems.
3. Compute the upper-bound of the expected number of inner iterations

$$\max_{k=1,\dots,k_{\text{out}}} \mathbb{E}[T_{\mathcal{M},G_k}(\varepsilon_k)] \leq k_{\text{in}},$$

Then, the expected iteration-complexity denoted Comp. is given by

$$\text{Comp} \leq k_{\text{in}} \times k_{\text{out}} \ .$$

Applications

Deterministic and Randomized Incremental Gradient methods

- Stochastic Average Gradient (SAG and SAGA) [Schmidt et al., 2013, Defazio et al., 2014a];
- Finito and MISO [Mairal, 2015, Defazio et al., 2014b];
- Semi-Stochastic/Mixed Gradient [Konečnỳ et al., 2014, Zhang et al., 2013];
- Stochastic Dual coordinate Ascent [Shalev-Shwartz and Zhang, 2012];
- Stochastic Variance Reduced Gradient [Xiao and Zhang, 2014].

But also, randomized coordinate descent methods, and more generally first-order methods with linear convergence rates.

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