

## Boundary Conditions for Young - van Vliet Recursive Filtering

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**Abstract**—Young & van Vliet designed computationally efficient methods for approximating Gaussian-based convolutions by running a recursive IIR filter forwards over the input signal, then running a second IIR filter backwards over the first filter’s output. However, to transition between the two filters, they used a suboptimal heuristic that produces significant amplitude and phase distortion for all points within about 3 standard deviations of the right-hand boundary. We derive a simple linear transition rule that eliminates this distortion.

**Index Terms**—Gaussian smoothing, bidirectional recursive filtering, boundary conditions.

### I. INTRODUCTION

Young & van Vliet (YvV) have developed computationally efficient forwards-backwards IIR recursions for Gaussian filters [1], Gaussian derivatives [2], and Gabor filters [3]. See [3] for their most recent design rules for Gaussians, and [4] for space-variant extensions and a performance comparison with other IIR Gaussian methods including Deriche’s original method [5], [6]. All of the YvV filters work forwards, recursively calculating a running sum  $u_t$  as a linear combination of the input signal  $i_t$  and the  $k$  previous  $u$  values, then work backwards calculating a running sum  $v_t$  as a linear combination of  $u_t$  and the  $l$  previously-calculated  $v$  values:

$$u_t = i_t + \sum_{j=1}^k a_j u_{t-j}, \quad t = 1 \dots n \quad (1)$$

$$v_t = u_t + \sum_{j=1}^l b_j v_{t+j}, \quad t = n \dots 1 \quad (2)$$

The final output is a scaled version of  $v_t$ , and  $\{a_{j,j=1\dots k}\}$  and  $\{b_{j,j=1\dots l}\}$  are suitably chosen filter coefficients. For Gaussians, YvV choose  $k=l$  and  $a=b$  [1], [3]. For other filters,  $i_t$  may be a linear transformation of the original input signal, *e.g.* a discrete derivative for derivative filters [1].

### II. THE PROBLEM WITH HEURISTIC BOUNDARY CONDITIONS

To complete the specification (1, 2), we must fix initial conditions for  $u$  near  $t=1$  and for  $v$  near  $t=n$ . For  $u$ , we can pretend that the signal existed and took some nominal constant value  $i_-$  (typically either 0 or  $i_1$ ) for all  $t < 1$ . The correct initialization at  $t=1$  is then to set all  $u_{1-j,j=1\dots k}$  to  $i_- / (1 - \sum_{j=1}^k a_j)$ , the steady state response to an infinite stream of  $i_-$ ’s. Similarly, if we could suppose that for all  $t > n$ ,  $u_t$  took some constant value  $u_+$ , the correct condition at  $t=n$  would be to set  $v_{n+j,j=1\dots l}$  to  $u_+ / (1 - \sum_{j=1}^l b_j)$ , the steady state response to an infinite stream of  $u_+$ ’s. YvV apparently do exactly this, with  $i_- = i_1$  and  $u_+ = u_n$  (*c.f.* [3] equations (20,21)). Another plausible choice for  $u_+$  would be  $i_+ / (1 - \sum_{j=1}^k a_j)$ , the steady state  $u$  resulting from an infinite stream of constant input values  $i_+$  above  $t=n$  (typically,  $i_+$  would be either  $i_n$  or 0).

Unfortunately, neither choice for  $u_+$  is correct. If the forwards filter were continued to  $t \gg n$  with input  $i_+$ , its output would decay smoothly from  $u_n$  to  $i_+ / (1 - \sum_{j=1}^k a_j)$  within a few standard deviations, and the corresponding backwards filter would take all elements of this “advance warning” signal into account when calculating its response. In fact, the forwards-backwards process *only* gives the

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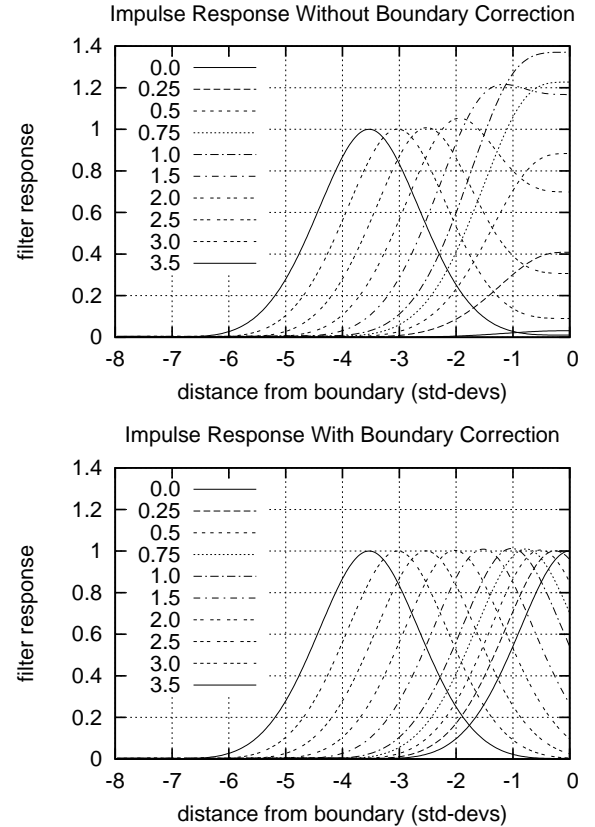


Fig. 1. Impulse responses for points at various numbers of standard deviations from the right boundary, for a YvV Gaussian filter, with the standard YvV boundary heuristic  $u_+ = u_n$  (top) and with our new boundary correction (14) (bottom). The corrected responses are much closer to the desired (truncated Gaussian) form.

correct overall impulse response when the full double recursion is run without truncation. Incorrect truncation causes significant amplitude and phase (geometric position) distortion for all points within about 3.5 standard deviations of the boundary. Fig. 1 illustrates the extent of the problem.

### III. DERIVATION OF LINEAR BOUNDARY CORRECTION

To correct for the effects of truncation, we notionally extend the forwards-backwards pass to  $t \rightarrow \infty$  assuming a constant input value  $i_+$  above  $t=n$ , and calculate the coefficients  $\{v_{n+j,j=1\dots l}\}$  that would result from this infinite extension, given  $i_+$  and the final forwards filter state  $\{u_{n-j,j=0\dots k-1}\}$ . The whole process is linear so the  $v$ ’s must be linear functions of the  $u$ ’s and  $i_+$ . First suppose that  $i_+ = 0$ . Gathering the  $u$ ’s,  $v$ ’s into running  $k, l$  vectors  $\mathbf{u}, \mathbf{v}$ , the forwards and backwards passes become:

$$\mathbf{u}_t = \mathbf{A} \mathbf{u}_{t-1} = \mathbf{A}^{t-n} \mathbf{u}_n \quad (4)$$

$$\mathbf{v}_t = \mathbf{I}_1 \mathbf{u}_t + \mathbf{B} \mathbf{v}_{t+1} \quad (5)$$

where

$$\mathbf{u}_t \equiv \begin{pmatrix} u_t \\ \vdots \\ u_{t-k+1} \end{pmatrix} \quad \mathbf{A} \equiv \begin{pmatrix} a_1 & \cdots & a_{k-1} & a_k \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad (6)$$

$$\mathbf{v}_t \equiv \begin{pmatrix} v_t \\ \vdots \\ v_{t+l-1} \end{pmatrix} \quad \mathbf{B} \equiv \begin{pmatrix} b_1 & \cdots & b_{l-1} & b_l \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad (7)$$

$$\mathbf{M} = \frac{1}{(1+a_1-a_2+a_3)(1-a_1-a_2-a_3) \cdot (1+a_2+(a_1-a_3)a_3)} \begin{pmatrix} -a_3a_1+1-a_3^2-a_2 & (a_3+a_1)(a_2+a_3a_1) & a_3(a_1+a_3a_2) \\ a_1+a_3a_2 & -(a_2-1)(a_2+a_3a_1) & -(a_3a_1+a_3^2+a_2-1)a_3 \\ a_3a_1+a_2+a_1^2-a_2^2 & a_1a_2+a_3a_2^2-a_1a_3^2-a_3^3-a_3a_2+a_3 & a_3(a_1+a_3a_2) \end{pmatrix} \quad (3)$$

and  $\mathbf{I}_1 = (1 \ 0 \ \dots \ 0)(1 \ 0 \ \dots \ 0)^\top$  is an  $l \times k$  matrix with a 1 in the top left corner and zeros elsewhere. Combining these equations for all  $t \geq n$ , we have  $\mathbf{v}_n = (\sum_{i=0}^{\infty} \mathbf{B}^i \mathbf{I}_1 \mathbf{A}^i) \mathbf{u}_n$ . We need to calculate the  $l \times k$  matrix  $\mathbf{M} \equiv \sum_{i=0}^{\infty} \mathbf{B}^i \mathbf{I}_1 \mathbf{A}^i$  that links the final forwards state  $\mathbf{u}_n$  to the initial backwards one  $\mathbf{v}_n$ . By its recursive definition:

$$\mathbf{M} = \mathbf{I}_1 + \mathbf{B} \mathbf{M} \mathbf{A} \quad (8)$$

Writing  $\mathbf{M} = \begin{pmatrix} m^1 \\ \vdots \\ m^l \end{pmatrix}$  by rows as a  $kl$ -element row vector  $\vec{\mathbf{M}} = (m^1, \dots, m^l)$  converts (8) to:

$$\vec{\mathbf{M}} = \vec{\mathbf{I}}_1 + \vec{\mathbf{M}} \begin{pmatrix} b_1 \mathbf{A} & \mathbf{A} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ b_{l-1} \mathbf{A} & \mathbf{0} & \dots & \mathbf{A} \\ b_l \mathbf{A} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \quad (9)$$

with solution ( $\mathbf{e}^1 = (1, 0, \dots, 0)$ ):

$$m^i = \mathbf{e}^1 \left( \mathbf{I} - \sum_{j=1}^l b_j \mathbf{A}^j \right)^{-1} \mathbf{A}^{i-1} \quad i = 1, \dots, l \quad (10)$$

Alternatively, writing  $\mathbf{M} = (\mathbf{m}_1, \dots, \mathbf{m}_k)$  by columns as a  $kl$  element column vector  $\mathbf{M}^\dagger$ , (8) becomes:

$$\mathbf{M}^\dagger = \mathbf{I}_1^\dagger + \begin{pmatrix} a_1 \mathbf{B} & \mathbf{B} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k-1} \mathbf{B} & \mathbf{0} & \dots & \mathbf{B} \\ a_k \mathbf{B} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \mathbf{M}^\dagger \quad (11)$$

with solution ( $\mathbf{e}_1 \equiv (1 \ 0 \ \dots \ 0)^\top$ ):

$$\mathbf{m}_1 = \left( \mathbf{I} - \sum_{j=1}^k a_j \mathbf{B}^j \right)^{-1} \mathbf{e}_1 \quad (12)$$

$$\mathbf{m}_i = \left( \sum_{j=i}^k a_j \mathbf{B}^{j-i+1} \right) \mathbf{m}_1 \quad i = 2, \dots, k \quad (13)$$

The  $\mathbf{M}$  of the ( $k=l=3$ ,  $\mathbf{a}=\mathbf{b}$ ) Gaussian filter recommended by YvV is given in (3) above.

Finally, to handle nonzero  $i_+$ , we can simply reduce to the  $i_+=0$  case by subtracting the constant- $u$  response  $u_+ = i_+ / (1 - \sum_{j=1}^k a_j)$  from each component of  $\mathbf{u}_n$ , apply  $\mathbf{M}$ , then add back the corresponding constant- $v$  response  $u_+ / (1 - \sum_{j=1}^l b_j)$  to get  $\mathbf{v}_n$ .

#### IV. SUMMARY OF METHOD

In summary, Young & Van Vliet recursive filters suffer from severe amplitude and phase distortion at the right boundary unless the backwards running coefficients are initialized from the forwards ones as follows, where  $\mathbf{M}$  is given by (8,3):

$$\begin{pmatrix} v_n \\ \vdots \\ v_{n+l-1} \end{pmatrix} = \mathbf{M} \begin{pmatrix} u_n - u_+ \\ \vdots \\ u_{n-k} - u_+ \end{pmatrix} + \begin{pmatrix} v_+ \\ \vdots \\ v_+ \end{pmatrix} \quad (14)$$

$$u_+ = \frac{i_+}{1 - \sum_{j=1}^k a_j} \quad v_+ = \frac{u_+}{1 - \sum_{j=1}^l b_j} \quad (15)$$

An implementation for 2D Gaussian image filtering will be made available on the author's web page. J.-M. Geusebroek has incorporated the technique in his IIR filters, available from his web site <http://www.science.uva.nl/~mark/downloads.html>

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