Surface Classification Using Conformal Structures

Xianfeng Gu
Department of CISE
University of Florida
Gainesville, FL 32611
gu@cise.ufl.edu

Shing-Tung Yau
Department of Mathematics
Harvard University
Cambridge, MA 02138
yau@math.harvard.edu

Abstract

3D surface classification is a fundamental problem in computer vision and computational geometry. Surfaces can be classified by different transformation groups. Traditional classification methods mainly use topological transformation groups and Euclidean transformation groups. This paper introduces a novel method to classify surfaces by conformal transformation groups. Conformal equivalent class is refiner than topological equivalent class and coarser than isometric equivalent class, making it suitable for practical classification purposes. For general surfaces, the gradient fields of conformal maps form a vector space, which has a natural structure invariant under conformal transformations. We present an algorithm to compute this conformal structure, which can be represented as matrices, and use it to classify surfaces. The result is intrinsic to the geometry, invariant to triangulation and insensitive to resolution. To the best of our knowledge, this is the first paper to classify surfaces with arbitrary topologies by global conformal invariants. The method introduced here can also be used for surface matching problems.

1. Introduction

3D surface classification and matching are fundamental problems in computer vision, computational geometry and computer aided geometric design. Recent developments in modelling and digitizing techniques have led to an increasing accumulation of 3D models. This has highlighted the need for efficient 3D objects searching techniques in large scale databases.

Many methods have been developed based on the topological and geometric features of the surfaces in order to describe shapes. In general, all the methods treat the surface as a two dimensional real manifold embedded in R^3, with the induced Euclidean metric structure.

In this paper, we view the surfaces from a completely novel viewpoint: treating them as Riemann surfaces with conformal structures, or namely one dimensional complex manifolds. A Riemann surface is a surface covered by holomorphic coordinate charts. The conformal structure is the complete invariants under conformal transformations and can be represented as matrices.

Compared to other surface classification methods, conformal classification has some advantages. Theoretically, conformal geometry has a sound foundation. Conformal equivalent classes are much refiner than the topological equivalent classes and much coarser than the isometric classes. Each topological equivalent class has infinite conformal equivalent classes, each conformal equivalent class has infinite isometric classes. The conformal structures are intrinsic to the geometry, independent of triangulation, insensitive to resolution and local features, and robust to noises. Also, conformal invariants are concise and efficient to compute, and can be used as search keys conveniently. Hence conformal classification is more suitable for practical surface classification problems.

Conformal invariants can also be used for general surface matching problems. In nature, it is highly unlikely for different shapes to share the same conformal structure. For many surface matching problems based on geometric features in the Euclidean space, conformal invariants can offer sufficient information to differentiate shapes. The computation of conformal invariants is much cheaper and more stable than computing geometric features.

To the best of our knowledge, although conformal structure is well known, we are the first group to systematically use it for surface classification problems.

We introduce previous work and the theoretic background in the following part of this section, followed by detailed explanation of the algorithms in Section two. Our surface classification method is introduced in Section three. Experimental results are reported in Section four. Finally, a brief summary and conclusion appears in Section five, followed by a discussion of topics for future work in Section six.

Proceedings of the Ninth IEEE International Conference on Computer Vision (ICCV 2003) 2-Volume Set
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1.1 Previous work

3D shape classification and recognition is a core problem in computer vision. Due to the apparent difference in nature, 2D shape classification methods can not be easily extended to 3D shape classification problems. To develop a 3D shape classification method, which makes use of 3D object topological and geometric features that is independent of tessellation and resolution, becomes desirable. Roughly, the current 3D shape classification methods fall into the following categories.

1. *Statistical properties based methods.* The simplest approach represents objects with feature vectors in a multidimensional space where the axes encode global geometric properties. Ankerst et al. [1] proposed shape histogram decomposing shells and sectors around a model's centroid. Osada et al. [12] represented shapes with probability distributions of geometric properties computed for points randomly sampled on an object's surface. However, these statistical methods are not discriminating enough to make subtle distinctions between shapes.

2. *Topology based methods.* Hilaga et al. [8] computed 3D shape similarity by comparing Multiresolutional Reeb Graphs(MRGs) which encodes the skeletal and topological structure at various levels of resolution. The MRG is constructed using a continuous function on the 3D shape, preferably a function of geodesic distance. These methods can not describe the geometric distinctions.

3. *Geometry based methods.* Novotni et al. [11] describe a method based on calculating a volumetric error between one object and a sequence of offset hulls of the other object. Tangelder et al. [15] represent the 3D shape by a signature representing a weighted point set. A shape similarity measurement based on weight transportation is used to compute the similarity between two shapes. Funkhouser et al. [3] developed a 3D matching algorithm that uses spherical harmonics to compute discriminating similarity measures. Kazhdan et al.[9] introduced a reflective symmetry descriptor that represents a measure of reflective symmetry for an arbitrary 3D model for all planes through the model's center of mass. These methods take into account of the embedding of the geometric shapes. The shape descriptors are represented as functions, inconvenient for searching. The classification is also too restrictive.

1.2 Theoretic background

The algorithms introduced in this paper are based on the theories of Riemann surfaces, especially the Abel-Jacobi theory as introduced in [4, 13]. We treat surfaces as Riemann surfaces, and compute their conformal structures represented as holomorphic one-forms and period matrices.

A *conformal map* is a map which only scales the first fundamental forms, hence preserving angles. If a mapping $f : M_1 \rightarrow M_2$ is conformal, where M_1 and M_2 are two surfaces, suppose (u^1, u^2) are local parameters and the Riemann metric (first fundamental form) of M_1 is $ds^2 = \sum_{ij} g_{ij} du^i du^j$, the metric of M_2 is $ds^2 = \sum ij \tilde{g}_{ij} du^i du^j$, then the induced metric $f^* \tilde{g}_{ij}$ satisfies

$$g_{ij}(u^1, u^2) = \lambda(u^1, u^2) f^* \tilde{g}_{ij}(u^1, u^2). \quad (1)$$

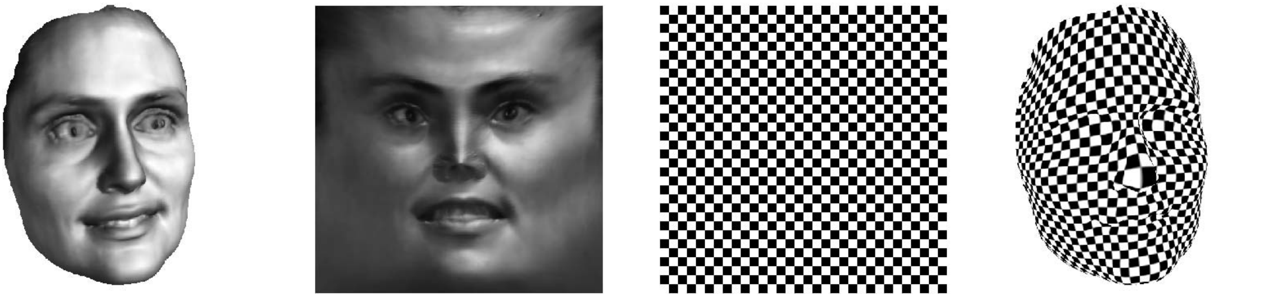
Figure 1 illustrates a conformal map from a real female face surface to a square. All the rights angles on the texture are preserved on the surface, which is shown in (c) and (d).

Two surfaces are called *conformal equivalent* if there exists a conformal diffeomorphism between them. Conformal equivalent surfaces share the same *conformal invariants*, which can be represented as a matrix.

Figure 2 shows two genus one surfaces. Although they are topologically equivalent, they are not conformal equivalent. Each torus can be cut open and conformally mapped to a planar parallelogram. The two tori can be conformally mapped to each other, if and only if one such parallelogam can be exactly matched to the other by translation, rotation and scaling. In other words, the shape of such parallelogram indicates the conformal equivalent class for the torus. We use the non-obtuse angle (right angle in this case) of the parallelogram and the length ratio between the two adjacent edges to represent the conformal invariants of the genus one surfaces. We call them *shape factors*. From (b) and (d), it is clear that the two tori have different shape factors, they are not conformal equivalent. Hence conformal classification is refiner than topological classification.

For higher genus surfaces, the conformal invariants are more complicated. Basically, handles of the surface can be cut open and conformally mapped to parallelograms with different shapes. The shape factors of all the handles indicate the conformal class. Figure 3 demonstrates a genus three surface, where each handle is conformally mapped to a parallelogram. The shapes of the three parallelograms are the conformal invariants. The rigorous representation of conformal invariants of a high genus surface, through *period matrices*, is explained below.

If a surface is mapped to the complex plane, and the mapping is conformal everywhere on the surface, then we call the complex gradient vector field of the mapping a *holomorphic one-form*. All the holomorphic one-forms on the surface form a real vector space, which we call *holomorphic*



(a) Original Surface (b) Conformal map to the plane (c) Checkerboard texture (d) Texture mapped surface

Figure 1. Conformal mapping. The original surface is a real human face (a), which is conformally mapped to a square (b). A checker board texture (c) is mapped back to the face. All the right angles on the texture are preserved on (d).

differentials. The dimension of the holomorphic differentials is equal to two times the surface's genus number.

All the oriented closed curves on the surface form a group in the sense that they can be duplicated, merged and reversed. Two closed curves are *homologous equivalent* if they together bound a 2D surface patch. The group of all the homologous equivalent classes is called the *homology group*. For surfaces, homology group can determine the topology.

Let M be a closed surface of genus g, and $B = \{e_1, e_2, \ldots, e_{2g}\}$ be an arbitrary basis of its homology group. We define the entries of the *intersection matrix* C of B as

$$c_{ij} = -e_i \cdot e_j \tag{2}$$

where the dot denotes the number of intersections, counting $+1$ when the direction of the cross product of the tangent vectors of e_i and e_j at the intersection point is consistent with the normal direction, -1 otherwise.

A holomorphic basis $B^* = \{\omega_1, \omega_2, \ldots, \omega_{2g}\}$ is defined to be dual of B if

$$Re \int_{e_i} \omega_j = c_{ij}. \tag{3}$$

Define matrix S as having entries

$$Im \int_{e_i} \omega_j = s_{ij}. \tag{4}$$

The matrix R defined as

$$CR = S \tag{5}$$

satisfies $R^2 = -I$, where I is the identity matrix. After H.Weyl [16] and C.L.Siegel [14], R is called the *period matrix* of M with respect to the homology basis B.

The matrices (R, C) determine the conformal equivalent class of M in the following sense: For two surfaces M_1 and M_2 with (R_1, C_1) and (R_2, C_2) respectively, M_1 and M_2 are conformal equivalent if and only if there exists an integer matrix N such that

$$N^{-1} R_1 N = R_2; N^T C_1 N = C_2. \tag{6}$$

We call (R, C) the *conformal structure* of M.

In the following sections, we will introduce a method to compute the shape factors for genus one surfaces, and (R, C) for higher genus surfaces, and use them to classify surfaces.

2 Computing Conformal Invariants

In this section, we will introduce a method to compute the homology group and holomorphic differential group of non-zero genus surfaces. We assume the surfaces are represented as triangular meshes. The method is improved upon the algorithm introduced in [6, 7, 2, 5].

Let K be a simplicial complex whose topological realization $|K|$ is homeomorphic to a compact 2-dimensional manifold. Suppose there is a piecewise linear embedding

$$F : |K| \to R^3.$$

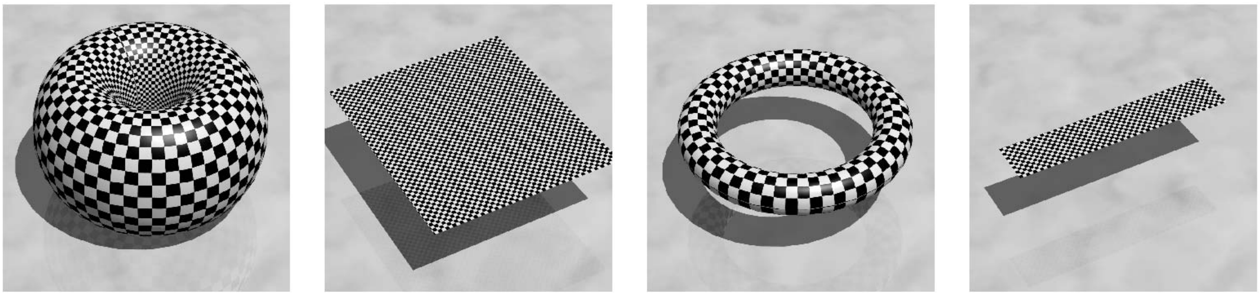
The pair (K, F) is called a triangular mesh and denoted as M.

2.1 Computing homology

Suppose M is a triangular mesh, u, v are vertices (0-simplices), we use $[u, v], [u, v, w]$ to represent its edges and faces(1-simplicies and 2-simplices). We define chain spaces as the following:

$$C_p K = \{\sum \alpha_i \sigma_p^i | \alpha_i \in Z\}, p = 0, 1, 2,$$

where σ_p^i's are p dimensional simplices in K. Therefore, the linear space $C_2 K$ is the space representing all the surface patches on M, $C_1 K$ is the the space representing all



(a) Torus one (b) Conformal map to a parallelogram (c) Torus two (d) Conformal map to a parallelogram

Figure 2. Topological equivalence but not conformal equivalence. The two tori (a) and (c) are topologically equivalent, but not conformally equivalent. Because they are conformally mapped to planar parallelograms with different shapes.

the curves on M, and C_0K is the space representing all the points on M. The boundary operators are linear mappings among these spaces $\partial_p : C_pK \to C_{p-1}K$:

$$\partial_p(\sum_i \alpha_i \sigma_p^i) = \sum_i \alpha_i \partial_p \sigma_p^i, p = 1, 2.$$

The boundary operators defined on each simplex are as follows:

$$\partial_2([u, v, w]) = [u, v] + [v, w] + [w, u]$$

where $[u, v, w]$ is a face, $[u, v]$, $[v, w]$ and $[w, u]$ are its three edges with consistent orientation. Therefore, ∂_2 is an operation that returns the boundary of a surface patch. ∂_1 is defined in a similar way $\partial_1([u, v]) = v - u$. ∂_2 and ∂_1 are linear operators and can be represented as integer matrices with elements 0, 1 or -1.

The kernel space of ∂_1 is the set of all closed curves, since closed curves do not have boundaries. The image space of ∂_2 is the set of all surface patch boundaries.

The homology group is defined as the quotient space in [10]

$$H_1(M, Z) = \frac{ker\partial_1}{img\partial_2}.$$

The homology bases are the eigenvectors of the kernel space of the linear operator $L : C_1K \to C_1K$:

$$L = \partial_1^T \partial_1 + \partial_2 \partial_2^T.$$

L is symmetric, the eigenvectors for the zero eigenvalue are the basis of $H_1(M, Z)$. Suppose $B = \{e_1, e_2, \cdots, e_{2g}\}$ is a set of homology basis, the intersection matrix C is a skew-symmetric matrix. In order to simplify the classification process, we can make the homology basis a canonical one $\widetilde{B}$, such that $\widetilde{B} = \{a_1, \cdots, a_g, b_1, \cdots, b_g\}$, $a_i \cdot b_i = +1$, and other intersection numbers are zeros. For any closed surfaces, such canonical homology basis always exists. Figure 4 illustrates such canonical homology bases for a genus 2 surface. Canonical homology basis is not unique, as shown in the figure. The intersection matrix of a canonical homology basis has a special format:

$$\widetilde{C} = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}, \quad (7)$$

where I_g is a $g \times g$ identity matrix, g is genus.

There exists an unitary integer matrix N(Its determinant is either 1 or -1), such that $NCN^T = \widetilde{C}$. Both C and $\widetilde{C}$ are congruent skew-symmetric matrices. C and $\tilde{C}$ can be diagonalized by orthonormal matrices U and V respectively, i.e. $C = U\Lambda U^T$ and $\widetilde{C} = V\Lambda V^T$, $\Lambda = diag\{J_1, J_2, \cdots, J_g\}$,

$$J_i = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, i = 1, 2, \ldots, g.$$

The N is simply $N = VU^T$. The canonical homology basis can be obtained by $\widetilde{B} = BN^T$. In the following discussion, we assume the homology bases are canonical ones.

2.2 Computing harmonic one-forms

We define the linear functional spaces of C_2K, C_1K and C_0K as C^2K, C^1K and C^0K respectively. In other words, C^2K is the set of all the linear functions defined on the surface patches, C^1K is the set of all linear functions defined on the curves on the surface. We can then define the coboundary δ_1 and δ_0 as the adjoint operator of ∂_2 and ∂_1, such that

$$\delta^p \omega(\sigma) = \omega \partial_{p+1}(\sigma), p = 0, 1, \quad (8)$$

where $\omega \in C^pK$, $\sigma \in C_{p+1}K$. Suppose $\omega \in C^1K$, if $\delta^1\omega \equiv 0$, then ω is called a *closed one-form*, and for any $[u, v, w] \in K$,

$$\delta\omega([u, v, w]) = \omega([u, v]) + \omega([v, w]) + \omega([w, u]) = 0. \quad (9)$$

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We use C^1K to represent tangential vector fields on M and associate an energy with each $\omega \in C^1K$:

$$E(\omega) = \frac{1}{2} \sum_{[u,v]\in K_1} k_{u,v} |\omega([u,v])|^2, \quad (10)$$

where $k_{u,v} = \frac{1}{2}(\cot\alpha + \cot\beta)$, α and β are the two angles against the edge $[u,v]$. $E(\omega)$ is called the *harmonic energy* of ω. A closed one-form which minimizes the harmonic energy is called a *harmonic one-form*. The Laplacian is a linear operator $\Delta : C^1K \to C^0K$,

$$\Delta\omega(u) = \sum_{[u,v]\in K} k_{u,v}\omega([u,v]). \quad (11)$$

Harmonic one-forms have zero Laplacian.

According to Hodge theory [13], all the harmonic one-forms form a real linear space, which can be treated as a dual space (linear functional space) of the homology group $H_1(M, Z)$. Given a homology basis $B = \{e_1, e_2, \cdots, e_{2g}\}$, we can compute a dual basis of the harmonic one-forms $\{\omega_1, \omega_2, \ldots, \omega_{2g}\}$ by the following linear system:

$$\left\{ \begin{array}{lll} \delta\omega_i & \equiv & 0 \\ \Delta\omega_i & \equiv & 0 \\ \int_{e_j} \omega_i & = & \delta_j^i \end{array} \right. , \quad (12)$$

where δ_j^i is the Kronecker symbol. This linear system can be solved by conjugate gradient methods efficiently.

2.3 Computing holomorphic one-forms

Holomorphic one-forms are the gradient fields of conformal maps, which can be formulated as $\omega + \sqrt{-1}\,{}^*\omega$, where ω and ${}^*\omega$ are harmonic one-forms, and ${}^*\omega$ is orthogonal to ω everywhere, i.e. ${}^*\omega = n \times \omega$, n is the normal field on M. ${}^*\omega$ is called the conjugate harmonic one-form of ω. In order to compute ${}^*\omega$, we construct a linear system based on the wedge product of closed one-forms. Given two closed one-forms τ_1, $\tau_2 \in C^1K$, we define the wedge product as the following linear operator $\wedge : C^1K \times C^1K \to C^2K$,

$$\tau_1 \wedge \tau_2([u,v,w]) = \frac{1}{6} \begin{vmatrix} \tau_1([u,v]) & \tau_1([v,w]) & \tau_1([w,u]) \\ \tau_2([u,v]) & \tau_2([v,w]) & \tau_2([w,u]) \\ 1 & 1 & 1 \end{vmatrix}. \quad (13)$$

Similarly, we can define the conjugate wedge product of τ_1 and τ_2, denoted as $\wedge^*$,

$$\tau_1 \wedge^* \tau_2([u,v,w]) = sMt^T, \quad (14)$$

where $s = (\tau_1([u,v]), \tau_1([v,w]), \tau_1([w,u]))$, $t = (\tau_2([u,v]), \tau_2([v,w]), \tau_2([w,u]))$, and

$$M = \frac{1}{24S} \begin{pmatrix} 2(l_2^2 + l_3^2) & l_1^2 + l_2^2 - l_3^2 & l_1^2 + l_3^2 - l_2^2 \\ l_1^2 + l_2^2 - l_3^2 & 2(l_3^2 + l_1^2) & l_2^2 + l_3^2 - l_1^2 \\ l_1^2 + l_3^2 - l_2^2 & l_2^2 + l_3^2 - l_1^2 & 2(l_1^2 + l_2^2) \end{pmatrix}, \quad (15)$$

l_i are the edge lengths, $||[u,v]|| = l_1$, $||[v,w]|| = l_2$, $||[w,u]|| = l_3$, and S is the area of face $[u,v,w]$.

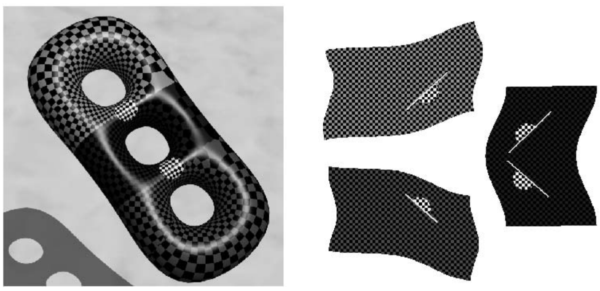
Given a harmonic one-form ω, then ${}^*\omega$ is still a harmonic one-form and satisfies the following linear equations

$$\int_M \omega_i \wedge ({}^*\omega) = \int_M \omega_i \wedge^* \omega, i = 1, 2, \ldots, 2g \quad (16)$$

where ω_i's are a set of basis of harmonic one-forms. Because ${}^*\omega$ is still a harmonic one-form, it can be represented as a linear combination of ω_i's, suppose ${}^*\omega = \sum_{i=1}^{2g} \alpha_i\omega_i$, then equation (16) becomes

$$\int_M \omega_i \wedge^* \omega = \sum_{j=1}^{2g} \alpha_i \int_M \omega_i \wedge \omega_j, i = 1, 2, \ldots, 2g \quad (17)$$

Given a harmonic one-form basis $\{\omega_1, \omega_2, \ldots, \omega_{2g}\}$, we can compute the conjugate harmonic one-forms ${}^*\omega_i$'s using 17, then $\{\omega_i + \sqrt{-1}\,{}^*\omega_i, i = 1, 2, \ldots, 2g\}$ is a basis of holomorphic one-forms.



(a) Genus 3 surface (b) Conformal mapping.

Figure 3. For higher genus surfaces, each handle can be conformally mapped to a parallelogram on the complex plane.

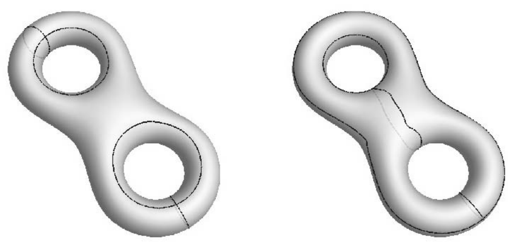


Figure 4. Canonical homology bases.

2.4 Computing period matrix

For a genus one surface M, there are two homology base curves e_1, e_2. Suppose e_1, e_2 only intersect at a point p. we can cut the surface open along e_1, e_2, then obtain a topological disk $\tilde{M}$. Then we choose one vertex to map to the origin of the complex plane, and integrate the holomorphic one-form on M'. Then $\tilde{M}$ is conformally mapped to the complex plane. Point p will be mapped to four corners of a parallelogram. We can compute the non-obtuse angle and adjacent edge length ratio of this parallelogram, which are the conformal invariants of M.

For a higher genus surface, suppose we have computed a canonical homology basis $\widetilde{B} = \{e_1, e_2, \dots, e_{2g}\}$, such that $e_i \cdot e_j = \delta_j^{i+g}, 1 \leq i \leq g < j \leq 2g$, δ_j^{i+g} is the Kronecker symbol, and constructed a dual holomorphic differential basis $B^* = \{\omega_1 + \sqrt{-1}\,^*\omega_1, \omega_2 + \sqrt{-1}\,^*\omega_2, \dots, \omega_{2g} + \sqrt{-1}\,^*\omega_{2g}\}$, then the matrices C and S have entries:

$$c_{ij} = \int_{e_i} \omega_j, s_{ij} = \int_{e_i} {}^*\omega_j. \tag{18}$$

Then R is computed as $R = C^{-1}S$. (R, C) are the conformal invariants.

2.5 Double covering

For surfaces with boundaries, we can convert them to closed ones by the so called *double covering* technique. Given a surface M with boundaries, we make a copy of M denoted as M', then reverse the orientation of M'. We simply glue M and M' together along their corresponding boundaries, the obtained surface $\widetilde{M}$ is a closed surface and called the *double covering* of M. We can then classify the surfaces with boundaries by the period matrices of its double covering.

All genus zero surfaces are conformal equivalent. It is impossible to differentiate them by their conformal structures directly. In practice, we can locate the critical points of their Gaussian curvature and remove them from the surface. The obtained surfaces are with boundaries and can be classified by using the double covering technique.

Figure 5 shows a genus zero example. Three holes are punched on the bunny surface, the bottom, and the tips of the ears. (d) illustrates a holomorphic one-form computed on the double covering, visualized by texture mapping a checkerboard pattern.

3 Surface Classification and Matching Method

Suppose M_1 and M_2 are two surfaces, the corresponding period matrices are (R_1, C_1) and (R_2, C_2) respectively. R_i can be decomposed as $P_i \Lambda_i P_i^{-1}$, where Λ_i is the Jordan norm form of R_i. If M_1 is conformal equivalent to M_2, then

$$\Lambda_1 = \Lambda_2 \tag{19}$$

and $N = P_1 P_2^{-1}$ is an integer matrix with determinant ± 1. Furthermore,

$$N^T C_1 N = C_2. \tag{20}$$

Equations (19) and (20) are the sufficient and necessary conditions to verify whether two surfaces are conformal equivalent. In our case, C_i's are canonical, matrices satisfying equation 20 are called symplectic matrices.

Then the surface classification problem is reduced to how to classify period matrices R under the integer symplectic matrix group. It has been proven that for genus g surfaces, the equivalent class of R is $6g - 6$ dimensional [4]. We will introduce a method to compute these $6g - 6$ parameters in our future work.

In practice, we use the sorted eigenvalues of R as the indices for surface indexing and matching.

4 Experiments Results

The algorithm is purely algebraic and easy to implement. The algorithm is intrinsic to the geometry, independent of triangulation and insensitive to resolution. The conformal structure is global and insensitive to local features and robust to noises. Figure 6 illustrates holomorphic one-forms, visualized by texture mapping a checkerboard image. The scaling of each texcel, and direction of isoparametric curves are consistent under different triangulation and resolution. Comparing (a) and (c), we can see that the resolutions and the triangulation are quite different. This shows the algorithm is intrinsic to the geometry and independent of the surface representation.

Table 1 shows the conformal invariants of the genus one surfaces illustrated in figure 7. By examining their shape factors, it is easy to verify that there are no two surfaces that are conformal equivalent.

mesh	angle (degree)	length ratio	vertices	faces
torus	89.987	2.2916	1089	2048
knot	85.1	31.150	5808	11616
knot2	89.9889	25.2575	2050	3672
rocker	85.432	4.9928	3750	7500
teapot	89.95	3.0264	17024	34048

Table 1. Conformal invariants of genus one surfaces.

The following are the period matrices R's for some genus two surfaces. All the intersection matrices C's are in the canonical form as equation (7).

The two hole torus mesh as shown in 8(a) has 861 vertices and 1536 faces. Its period matrix is

$$\begin{pmatrix} -1.475e-3 & 4.840e-4 & 4.501e-1 & 2.132e-2 \\ 4.858e-4 & -1.439e-3 & 2.132e-2 & 4.501e-1 \\ -2.260e+0 & 1.090e-1 & 1.476e-3 & -4.858e-4 \\ 1.090e-1 & -2.260e+0 & -4.840e-4 & 1.439e-3 \end{pmatrix} \quad (21)$$

The vase model shown in 8(b) has 1582 vertices and 2956 faces. Its period matrix is

$$\begin{pmatrix} 1.053e-3 & -8.838e-6 & 4.479e-1 & 2.127e-2 \\ -1.080e-4 & -1.031e-3 & 2.127e-2 & 4.042e-1 \\ 2.309e+0 & 1.241e-1 & 1.053e-3 & -1.080e-4 \\ -1.241e-1 & -2.564e+0 & 8.851e-6 & 1.031e-3 \end{pmatrix} \quad (22)$$

The flower model shown in 8(c) has 5112 vertices and 10000 faces. Its period matrix is

$$\begin{pmatrix} 6.634e-3 & -1.950e-3 & 2.861e-1 & -6.076e-2 \\ -1.909e-3 & 7.091e-3 & -6.076e-2 & 2.497e-1 \\ -3.768e+0 & -9.111e-1 & -6.634e-3 & 1.909e-3 \\ -9.111e-1 & -4.303e+0 & 1.950e-3 & -7.091e-3 \end{pmatrix} \quad (23)$$

The knotty bottle model shown in 8(d) has 15000 vertices and 30000 faces. Its period matrix is

$$\begin{pmatrix} -1.911e-2 & 2.757e-3 & 5.617e-2 & -1.001e-3 \\ 1.213e-3 & -9.294e-2 & -1.003e-3 & 5.699e-2 \\ -1.792e+1 & -4.829e-1 & 1.912e-2 & -6.224e-4 \\ -4.817e-1 & -1.819e+1 & -3.355e-3 & 9.295e-2 \end{pmatrix} \quad (24)$$

By checking the conditions of equations (19) and (20), it can be verified easily that all the surfaces above belong to different conformal equivalent classes.

We tested our algorithm on other complex models scanned from real models, the highest genus is 7 and the biggest surface is with hundreds of thousands of faces. The computational procedure is stable. We also retriangulated several surfaces, and compared the computing results, which are very close. For example, we computed the shape factors of the teapot surfaces with different resolutions as shown in figure 6, the high resolution shape factors are $(89.95, 3.0264)$, the low resolution shape factors are $(89.98, 3.0936)$.

5. Summary and Conclusions

This paper introduces a surface classification method based on the Riemann surface theories. All surfaces can be classified by the conformal transformation group and their conformal invariants can be represented by period matrices. The method is intrinsic to the geometry, independent of triangulation and insensitive to resolution. The conformal invariants are global features of surfaces, hence they are robust to noises. The conformal equivalent classification is refiner than topological classification and coarser than isometric classification, making it suitable for surface classifications and matching.

6 Future Work

In the future, we will test our algorithm using larger scale geometric databases. We also would like to explore ways to improve efficiency in computing harmonic one-forms, which is the most time consuming step of the current process. We also would like to generalize our current algorithm to non-manifold surfaces and implicit surfaces.

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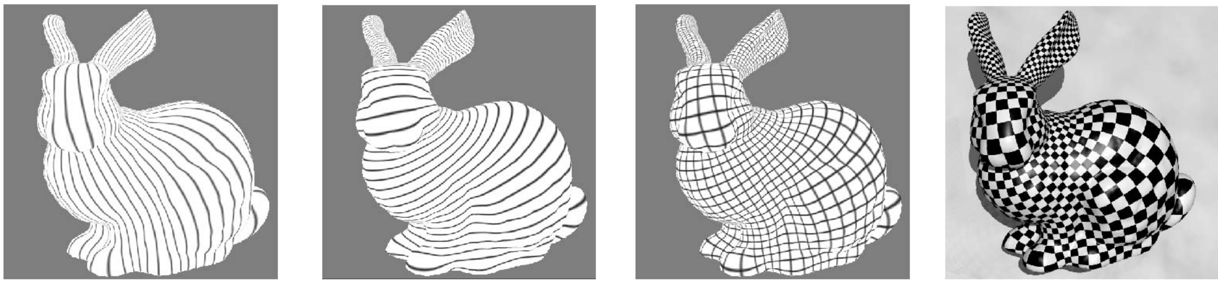
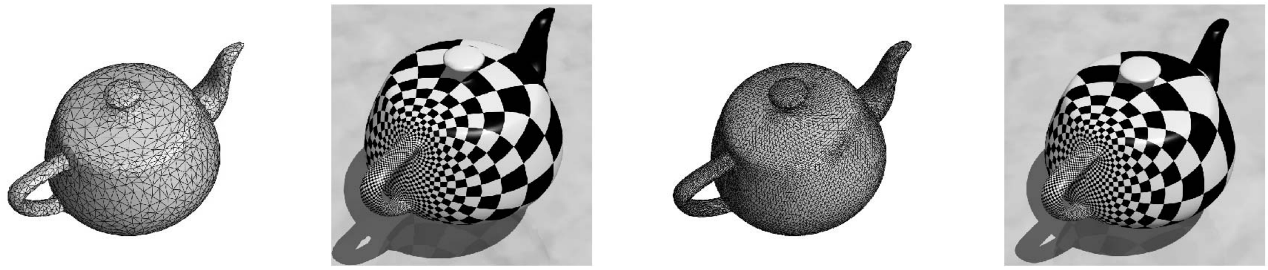


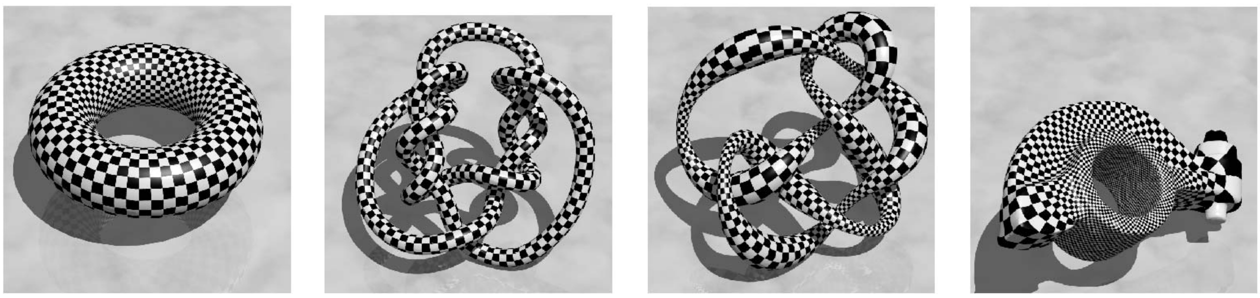
(a) Integration curve of the real part of (c) (b) Integration curve of the Imaginary part of (c) (c) Gradient field of a conformal map (d) Texture mapping generated by (c)

Figure 5. Holomorphic 1-form is a complex gradient field of a conformal map from the surface to the complex plane. (d) visualizes the holomorphic 1-form by a texture mapping.



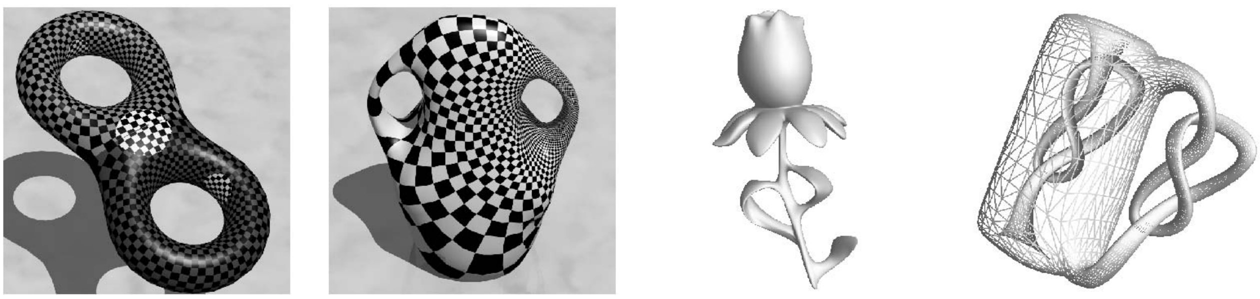
(a) Surface with 4K faces (b) Holomorphic 1-form of (a) (c) Surface with 34K faces (d) Holomorphic 1-form of (c)

Figure 6. Conformal structure is only dependent on geometry, independent of triangulation and insensitive to resolution.



(a) A torus surface (b) A knot surface (a) (c) Another knot surface (d) A rocker

Figure 7. Genus one surfaces with different conformal structures.



(a) A two hole torus surface (b) A vase surface (c) A rose surface (d) A knotty surface

Figure 8. Genus two surfaces with different conformal structures.