“Perspective Shape from Shading” and Viscosity Solutions

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Abstract

This article proposes a solution of the Lambertian shape from shading (SFS) problem in the case of a pinhole camera model (performing a perspective projection). Our approach is based upon the notion of viscosity solutions of Hamilton-Jacobi equations. This approach allows us to naturally deal with nonsmooth solutions and provides a mathematical framework for proving correctness of our algorithms. Our work extends previous work in the area in three aspects. First, it models the camera as a pinhole whereas most authors assume an orthographic projection (see [15] for a panorama of the SFS problem up to 1989 and [29, 17] for a recent survey), thereby extending the applicability of shape from shading methods to more realistic images. In particular it extends the work of [24] and [26]. Second, by adapting the brightness equation to the perspective problem, we obtain a new partial differential equation (PDE). Results about the existence and uniqueness of its solution are also obtained. Third, it allows us to come up with a new approximation scheme and a new algorithm for computing numerical approximations of the ”continuous” solution as well as a proof of their convergence toward that solution.

1. Introduction

SFS has been a central problem in the field of computer vision since the early days. The problem is to compute the three-dimensional shape of a surface from the brightness variations in a grey level image of that surface. The work in our field was pioneered by Horn who was the first to pose the problem as that of finding the solution of a nonlinear first-order PDE called the brightness equation [13]. Later on, various approaches have been proposed: the book [15] contains a very nice panorama of the research in SFS up to 1989; for a more recent overview, see [29, 17]. Despite the richness of the literature in this area, most SFS algorithms have been developed under the assumption of orthographic projection. Few SFS approaches consider the perspective projection problem (i.e consider a pinhole camera model as opposed to a simple affine model). Penna [22, 21] proposes a local method using geometrical properties. His formulation of the problem leads to solve a system of algebric equations. Lee and Kuo [18] present a variational approach. They minimize a cost functional based on a local linear approximation of the reflectance map. Hasegawa and Tozzi [12] suggest to combine SFS technique with photogrammetric technique to reconstruct the surface and to calibrate the camera. Their method consists in solving large systems of linear equations and seems to be suitable only for small images.

By the perspective projection hypothesis, these three approaches extend the applicability of SFS methods to more realistic images: we can recover the shapes of objects which are located near the camera (see [28] for an example of application).

In this article, we assume that the camera performs a perspective projection of the scene. We propose a global method and we do not need to linearize the reflectance map. Also, in the previous three approaches, the authors use a 3D coordinate system attached to the scene. We find it a lot simpler to use the camera coordinate system. In section 2, this choice of coordinates allows us to formulate the problem as a new PDE.

A very important aspect of the SFS problem is the question of the existence and uniqueness of a solution. This question as well as those related to the convergence of numerical schemes for computing the solutions became central in the last decade of the 20th century. For example, the papers of Bruss [5], Brooks [4], Horn [14], and Durou [9, 10], show the difficulty of the questions of existence and uniqueness. The first results related to the convergence of the numerical approximations have been presented by Dupuis and Oliensis [8] and P.-L. Lions, Rouy and Tourin [26, 20]. These results have been generalized by the recent papers of Falcone [11] and of Prados, Faugeras and Rouy [24]. Let us mention here that all of these works only deal with the simplest version of the SFS problem (with orthographic projection). In this article, we deal with the same

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questions in the framework of "perspective SFS" problem. Let us also remark that the papers of Penna [22, 21], Lee and Kuo [18], and Hasegawa and Tozzi [12] do not deal at all with these questions.

The paper is organized as follows. In section 2 we adapt the brightness equation to the perspective problem. Also, we obtain a new PDE. In section 3 we develop original results about existence and uniqueness of a viscosity solution of the perspective SFS equation (3). In section 4, we present an approximation scheme and numerical algorithm for computing numerical approximations of the solution of the "perspective SFS" problem. Also we give conditions for ensuring that the scheme and algorithm (which are consistent and monotonous) converge toward the viscosity solution of the "perspective SFS" PDE (3). In section 5 we show some experimental results on synthetic images and conclude in section 6.

2. Original adaptation of the brightness equation to "Perspective SFS Problem"

We assume that the camera performs a perspective projection of the scene, that the scene is illuminated by a single point source at infinity, that its reflectance is Lambertian and its albedo constant and equal to 1. We also assume that there are no shadows and no occluding boundaries and that the distance from the camera to the scene is known on the boundary of the image. Admittedly, these hypotheses may appear a bit restrictive and we will describe in an another paper how to remove some of these constraints. The camera is characterized and represented by the retinal plane R and by the optical center as shown in figure 1. We denote f the focal distance. The scene is represented by a surface S. We assume that S can be explicitly parameterized by using a function u defined on $\overline{\Omega}$:

$$S = \left\{ u(x_1, x_2). \begin{pmatrix} x_1 \\ x_2 \\ f \end{pmatrix}; \quad (x_1, x_2) \in \overline{\Omega} \right\}.$$

We also suppose that the surface is visible (in front of the retinal plane); so u verifies $\forall (x_1, x_2) \in \overline{\Omega}, \quad u(x_1, x_2) \geq 1$.

We note $\mathbf{n}(x)$ the normal vector to the surface S at point $u(x)\cdot(x,\ f)$ for all $x = (x_1, x_2)$ in Ω, $\mathbf{L} = (\alpha, \beta, \gamma)$ the unit vector representing the direction of the light source ($\gamma < 0$), $\mathbf{l} = (\alpha, \beta)$, and I the image intensity. The image is modelled as a function from the closure $\overline{\Omega}$ of an open set Ω of $\mathbb{R}^2$ into the closed interval $[0, 1]$, $I : \overline{\Omega} \longrightarrow [0, 1]$. For all $x \in \overline{\Omega}$ the intensity $I(x)$ is the brightness obtained at the retinal point $(x_1, x_2,\ f) \in R$. The lambertian hypothesis implies:

$$I(x) = \frac{\mathbf{n}(x) \cdot \mathbf{L}}{|\mathbf{n}(x)|}. \tag{1}$$

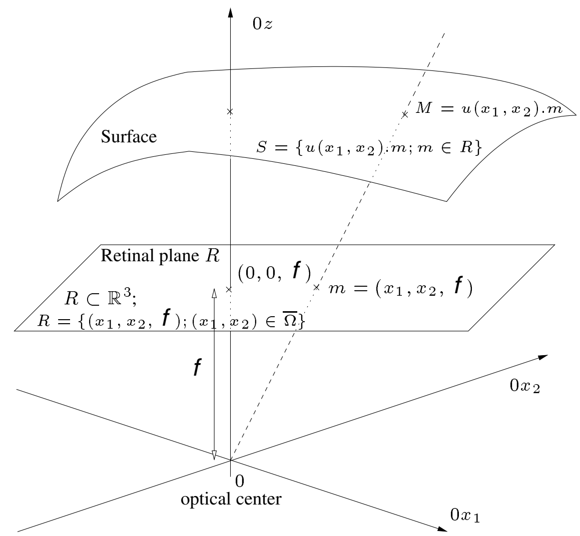


Figure 1. $I(x_1, x_2)$ **is the brightness obtained at the point** $(x_1, x_2,\ f)$**.**

An explicit expression for $\mathbf{n}(x)$ can easily be obtained through differential calculus,

$$\mathbf{n}(x) = u(x) \begin{pmatrix} f \nabla u(x) \\ -(u(x) + x \cdot \nabla u(x)) \end{pmatrix}.$$

The irradiance equation (1) then becomes the following PDE:

$$I(x) = \frac{f\ \mathbf{l} \cdot \nabla u(x) - \gamma\ (x \cdot \nabla u(x) + u(x))}{\sqrt{f^2 |\nabla u(x)|^2 + (x \cdot \nabla u(x) + u(x))^2}} \tag{2}$$

Since $u \geq 1$ and equation (2) is homogeneous in $\nabla u(x)$ and $u(x)$, we can simplify this equation by the change of variables $v = ln(u)$. Thus the "perspective SFS problem" consists in solving the original PDE:

$$I(x)\sqrt{f^2|\nabla v|^2 + (x \cdot \nabla v + 1)^2} - (\ f\ \mathbf{l} - \gamma x) \cdot \nabla v + \gamma = 0. \tag{3}$$

Let us emphasize that, unlike Penna [22, 21], Lee and Kuo [18] or Hasegawa and Tozzi [12], our formalism yields an explicit PDE. Also, this formulation allows to prove existence and uniqueness results for the "perspective SFS" problem.

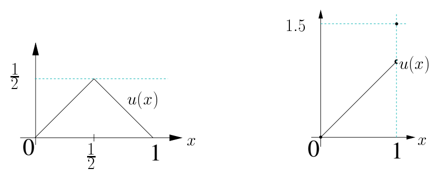
3. Existence and Uniqueness Results for "Perspective SFS Problem"

The notion of viscosity solutions of Hamilton-Jacobi PDEs has been introduced by Crandall and Lions [19, 7, 6]. For more recent results, see [1] or [2]. For an intuitive approach connected to computer vision, see [25].

The notion of viscosity solutions was first used to solve SFS problems by Lions, Rouy and Tourin [26, 20] in the 90s. Their work was based upon the notion of *continuous* viscosity solution. The continuous viscosity solutions are PDE solutions in a weak sense. In particular, they are not necessarily differentiable and can have edges. This notion allows to define a solution of a PDE which does not have classical solutions. For example, the equation

$$|\nabla u(x)| = 1 \text{ for all } x \text{ in }]0,1[\tag{4}$$

with $u(0) = u(1) = 0$, does not have classical solutions (Rolles theorem) but has a continuous viscosity solution (see figure 2-a)). Let us emphasize that continuous viscosity solutions are continuous (on the closure of the set where it is defined) and that a solution in classical sense is a viscosity solution.



a) b)

Figure 2. a) Continuous viscosity solution of (4) with $u(0) = u(1) = 0$; b) discontinuous viscosity solution of (4) with $u(0) = 0$ and $u(1) = 1.5$.

By the previous section, the "perspective SFS problem" consists in solving the PDE (3). Since the PDE (3) does not depend on v, we impose Dirichlet boundary conditions for ensuring uniqueness :

$$\forall x \in \partial\Omega,\ u(x) = \varphi(x),\ \varphi \text{ being continuous on } \partial\Omega. \tag{5}$$

In [23], we prove that the theory of continuous viscosity solutions applies to equation (3) with the Dirichlet boundary conditions (5). In the particular case where the light direction corresponds to the optical axis of the camera ($\mathbf{L} = (0, 0, -1)$), we prove that

∘ if I is Lipschitz continuous,

∘ if for all x in $\overline{\Omega}$, $I(x) > \frac{|x|}{\sqrt{f^2+|x|^2}}$,

∘ and if φ verifies the classical compatibility condition,

then the PDE (3) has a continuous viscosity solution. Also, by using the uniqueness theorem of Rouy and Tourin [26], we show that if furthermore $I(x) < 1$ for all $x \in \Omega$, then there exists at most one continuous viscosity solution.

Recently, Prados, Faugeras and Rouy [24] have proposed to use the more general idea of *discontinuous* viscosity solutions (see [2]) in order to get away from the compatibility conditions (on the boundary conditions) that are necessary for the existence of continuous solutions. Thus, with this notion of discontinuous solutions, we can recover solutions of SFS even with large errors on the boundary conditions (which we suppose known). For example, the previous equation (4) with $u(0) = 0$, $u(1) = 1.5$ does not have continuous viscosity solutions, but have a discontinuous viscosity solution (see figure 2-b)). Also let us remark that a "discontinuous viscosity solution" *can* hold discontinuities and that a continuous viscosity solution is a discontinuous viscosity solution.

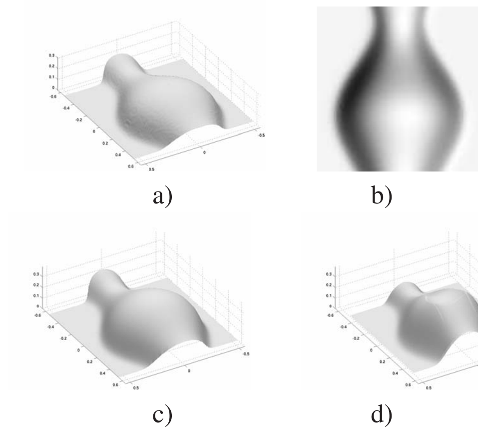


Figure 3. Results for an image generated by a smooth surface: a) original surface, b) original image ($\mathbf{l} = (0.2, 0.2)$, $r = 0.4$, size=128×128), c) surface reconstructed from b) by the "perspective algorithm": $n \approx 1000$, $\epsilon_1 = 2.3\%$, $\epsilon_2 = 2.4\%$ $\epsilon_\infty = 2.8\%$; d) surface reconstructed from b) by the "orthographic algorithm": $n \approx 1000$, $\epsilon_1 = 15.8\%$, $\epsilon_2 = 16.1\%$ $\epsilon_\infty = 22.3\%$;

As for the continuous case, we prove existence and uniqueness results for the discontinuous viscosity solutions of the "perspective SFS" equation (3). By using the "Legendre transform" [19] and differential calculus, we rewrite equation (3) as the supremum PDE $\sup_{a \in B(0,1)}\{-f(x,a) \cdot \nabla u - l(x,a)\} = 0$
where, for $x \neq 0$:

· $f(x,a) = -[B_x a + w_x]$ and

· $l(x,a) = -[R_x\sqrt{1-|a|^2} + V_x \cdot a + \gamma]$ with

· $B_x = \frac{I(x)}{|x|}\begin{pmatrix} f x_2 & \sqrt{f^2+|x|^2}x_1 \\ -f x_1 & \sqrt{f^2+|x|^2}x_2 \end{pmatrix}$, $R_x = \frac{I(x)\, f}{\sqrt{f^2+|x|^2}}$,

· $w_x = \gamma x - f\mathbf{1}$, and $V_x = I(x)(0, \frac{|x|}{\sqrt{f^2+|x|^2}})$.

For $x = 0$:
$f(x,a) = f(\mathbf{1} - I(0)a)$ and $l(x,a) = -[I(0)\sqrt{1-|a|^2} + \gamma]$.
Thanks to this other representation, we prove that as soon as I is Lipschitz continuous on $\overline{\Omega}$, there always

exists a discontinuous viscosity solution of (3) with any continuous boundary conditions. If furthermore $\frac{|x|}{\sqrt{f^2+|x|^2}} < I(x) < 1$, for all $x \in \overline{\Omega}$ (in the case where $\mathbf{L} = (0, 0, -1)$), then the strong uniqueness theorem of Barles [2] applies; and so there exists at most one discontinuous viscosity solution of (3).

In practice, I can reach the value 1 but in this case, we lose uniqueness. For the orthographic version of the SFS problem, the loss of uniqueness is completely characterized by [20, 26, 24]. As for the orthographic version, to recover uniqueness, we only need to ignore the set $\{I = 1\}$ and work in the open set $\Omega' = \Omega - \{I = 1\}$ instead of Ω.

For more details about these statements and these theorems, see our technical report [23].

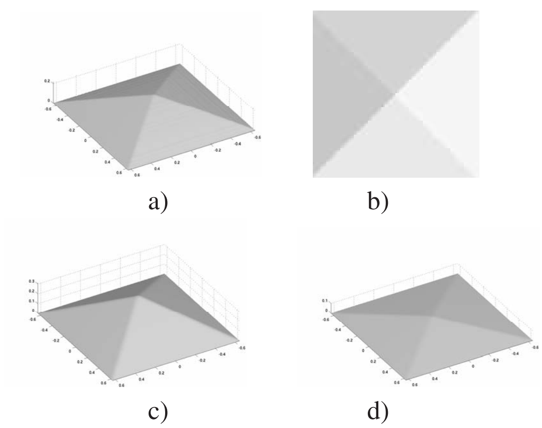


Figure 4. Results for an image generated by a pyramidal surface: a) original surface, b) original image (l $= (0.2, 0.2)$, $r = 0.47$, size$=$ 100×100), c) surface reconstructed from b) by the "perspective algorithm": $n = 76$, $\epsilon_1 =$ 0.25%, $\epsilon_2 = 0.26\%$ $\epsilon_\infty = .37\%$; **d) surface reconstructed from b) by the "orthographic algorithm":** $n = 83$, $\epsilon_1 = 49.1\%$, $\epsilon_2 = 49.7\%$ $\epsilon_\infty = 52.7\%$;

4. Approximation Scheme and Numerical Algorithm for Solving "Perspective SFS"

In the previous section, we have shown that the "perspective SFS" equation (3) can be rewriten as follows:

$$\begin{cases} \sup_{a \in A}\{-f(x,a) \cdot \nabla u(x) - l(x,a)\} = 0 & \text{if } x \in \Omega, \\ u(x) - \varphi(x) = 0 & \text{if } x \in \partial\Omega. \end{cases} \tag{6}$$

In this section we develop a method to obtain numerical approximations of the viscosity solutions of this equation.

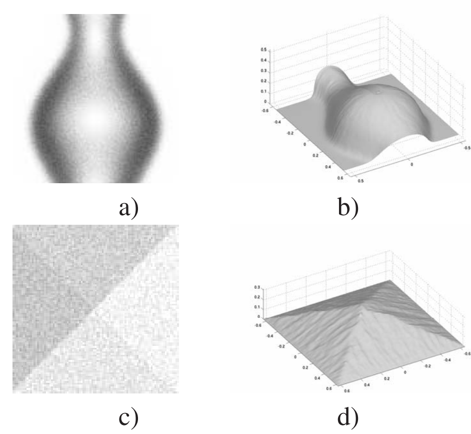


Figure 5. Results for the images 3-b) and 4-b) corrupted by a uniformly distributed noise: a) noisy image of the vase; b) surface reconstructed from a): $n \approx 2000$, $\epsilon_1 = 3.5\%$, $\epsilon_2 = 3.9\%$ $\epsilon_\infty = 6.2\%$; **c) noisy image of the pyramid; d) surface reconstructed from c):** $n \leq 100$, $\epsilon_1 = 0.5\%$, $\epsilon_2 = 0.9\%$ $\epsilon_\infty = 1.0\%$;

4.1. Approximation Scheme and Convergence

For all mesh size $\rho = (\Delta x_1, \Delta x_2)$, let us consider the finite difference scheme $S(\rho, x, u(x), u) = 0$ with

$$S(\rho, x, t, u) = \max_{s_1, s_2 = \pm 1} S_{s_1,s_2}(\rho, x, t, u) \tag{7}$$

where $S_{s_1,s_2}(\rho, x, t, u) =$

$$\sup_{a \in A_{s_1,s_2}} \left\{ -f(x,a) \cdot \begin{pmatrix} s_1 \frac{t - u(x - s_1 \Delta x_1 e_1)}{\Delta x_1} \\ s_2 \frac{t - u(x - s_2 \Delta x_2 e_2)}{\Delta x_2} \end{pmatrix} - l(x,a) \right\},$$

$A_{s_1,s_2} = \{a \in A | -f_1(x,a)s_1 \geq 0 \text{ and } -f_2(x,a)s_2 \geq 0\}$.

Using Barles and Souganidis definitions (see [3]), we prove that the scheme (7) is always monotonous and consistent with equation (3). Also, this scheme is stable as soon as it has a subsolution and the equation (3) is coercive in ∇v uniformly with respect to x. Thus with the same hypotheses, as soon as the strong uniqueness property is true, we prove that the solution u^ρ of the scheme (7) converges toward the unique discontinuous viscosity solution of "perspective SFS" equation when $\rho \to 0$.

For more details and for the proofs of these statements, see [23].

4.2. Numerical Algorithm and Convergence

In the previous section, we designed a scheme and described conditions which involve the convergence of its solutions u^ρ toward the unique viscosity solution of (3). We

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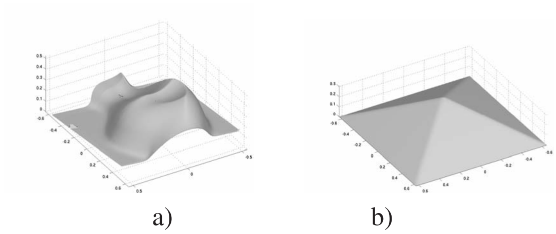

Figure 6. Smooth and pyramidal surfaces of figure 3 and 4 reconstructed with an $5°$ error on the light parameter $\mathbf{L}$: a) $n \approx 2000$, $\epsilon_1 = 24.4\%$, $\epsilon_2 = 26.3\%$ $\epsilon_\infty = 35.9\%$; **b)** $n \leq 100$, $\epsilon_1 = 1.50\%$, $\epsilon_2 = 1.58\%$, $\epsilon_\infty = 2.64\%$.

now describe an algorithm that computes an approximation of u^ρ, for each value of ρ.

For a fixed $\rho = (\Delta x_1, \Delta x_2)$, let us note, for (i,j) in $\mathbb{Z}^2$, $x_{ij} = (i\Delta x_1, j\Delta x_2)$, $\mathcal{X} := \{x_{ij}\}_{(i,j)\in\mathbb{Z}^2}$,$Q^\rho = \Omega^\rho \cap \mathcal{X}$ and $Q = \overline{\Omega} \cap \mathcal{X}$. The algorithm consists of the following computation of the sequence of values U^n_{ij}, $n \geq 0$ (U^n_{ij} being an approximation of $u^\rho(x_{ij})$).

Algorithm 1 *1. Initialisation ($n = 0$): $U^0_{ij} = u_0(x_{ij})$.*

2. *Choice of a pixel $x_{ij} \in Q^\rho$ and modification of U^n_{ij}: We choose U^{n+1} such that $\forall (k,l) \neq (i,j)$, $U^{n+1}_{kl} = U^n_{kl}$ and $S(\rho, x_{ij}, U^{n+1}_{ij}, U^n) = 0$.*
3. *Choose the next pixel $x_{ij} \in Q^\rho$ in such a way that all pixels of Q^ρ are regularly visited and go back to 2.*

In the previous algorithm, u_0 is a subsolution of the scheme. We prove that the algorithm is well-defined and the constructed sequence U^n is increasing and converges when $n \to +\infty$ towards the solution u^ρ of the scheme (7). Let us recall that with adequate hypotheses, the solution u^ρ of the scheme (7) converges toward the unique discontinuous viscosity solution of (3).

5. Experimental Results

We have applied the method presented in section 4 for computing numerical approximations of the general "perspective SFS" equation (3) with Dirichlet boundary conditions. We have tested our algorithm with synthetic images generated by shapes with several levels of regularity e.g. C^∞ (a smooth surface: the classical vase presented in [29] that we have suitably smoothed; see figures 3), or C^0 (a pyramid, see figures 4), to demonstrate the ability of our method to deal with smooth and nonsmooth objects. Since we deal with synthetic surfaces, we can use the exact boundary conditions to compute the solutions.

In all results, the parameters are n, the number of iterations, ϵ_1, ϵ_2 and ϵ_∞ the relative errors of the computed surface measured according to the L_1, L_2 and L_∞ norms, respectively, r the ratio of the focal and object distances (the object distance is the mean distance of the points of the surface to the optical center).

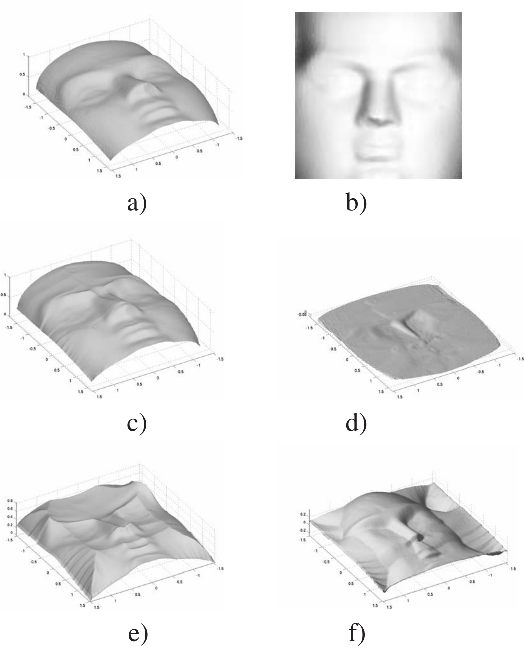

Figure 7. Results for an image generated by the classical Mozart's face: a) original surface, b) original image ($\mathbf{l} = (0.1, 0.1)$, $r \approx 0.6$, **size**$= 128 \times 128$**), c) surface reconstructed from b) by the "perspective algorithm", d) the associate error surface:** $n \approx 4000$, $\epsilon_1 = 3.0\%$, $\epsilon_2 = 3.6\%$ $\epsilon_\infty = 4.4\%$**; e) surface reconstructed from b) by the "orthographic algorithm" and f) the associate error surface:** $n \approx 5600$, $\epsilon_1 = 14.8\%$, $\epsilon_2 = 17.1\%$ $\epsilon_\infty = 30.3\%$**;**

In all cases we show the original object, the input image, the surface reconstructed by the new "perspective algorithm" and the surface reconstructed by the "orthographic algorithm" proposed in [24]. We notice that, as soon as the ratio r is smaller than 1/2, the "orthographic algorithm" [24] produces important errors whereas the quality of the results obtained by the "perspective algorithm" remains very good.

We also demonstrate the stability of our method with respect to two types of errors. The first type is image intensity errors due to noise. Uniformly distributed noise has been added to the input images and the corresponding reconstructed surfaces are shown (figure 5). The second type of error is due to an incorrect knowledge of the direction of illumination $\mathbf{L}$. The figure 6 shows the reconstructed surfaces with a $5°$ error on the parameter $\mathbf{L}$.

As seen from these figures, our algorithm seems to be quite robust, not only to intensity noise (see figures 5), but also to inaccuracies in the estimation of the direction of the light source **L** (see figure 6).

The pyramid example shows the remarkable ability of the numerical scheme to deal with functions which are only continuous. This example also shows the convergence of our algorithm with discontinuous images: we hope to extend shortly our theory to discontinuous images (see recent results of [27, 16]).

We have also tested our algorithm on more complicated images; for example we have tested it with part of the classical Mozart's face presented in [29]. Figure 7 shows the results obtained with $\mathrm{l} = (0.1, 0.1)$, $f = 1$, and $r = 0.6$.

6. Conclusion

We have proposed a new method for recovering the shape of Lambertian objects from shaded images under perspective projection. This approach is based on a rigorous mathematical analysis; we have given results about the existence and uniqueness of a viscosity solution, provided an approximation scheme and a numerical algorithm for computing this solution. Finally, we have proved the convergence (with adequate hypotheses) of our numerical approximations toward the viscosity solution of our PDE.

We are extending our approach for recovering non Lambertian surfaces and for removing the requirement for boundary conditions.

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