Stochastic gradient methods for machine learning

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Joint work with Eric Moulines, Nicolas Le Roux and Mark Schmidt - January 2013

Context
Machine learning for "big data"

- **Large-scale machine learning**: **large p, large n, large k**
  - p : dimension of each observation (input)
  - k : number of tasks (dimension of outputs)
  - n : number of observations

- **Examples**: computer vision, bioinformatics, signal processing

- **Ideal running-time complexity**: $O(pn + kn)$

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- **Going back to simple methods**
  - Stochastic gradient methods (Robbins and Monro, 1951)
  - Mixing statistics and optimization
  - It is possible to improve on the sublinear convergence rate?

Outline

- **Introduction**
  - Supervised machine learning and convex optimization
  - Beyond the separation of statistics and optimization

- **Stochastic approximation algorithms** (Bach and Moulines, 2011)
  - Stochastic gradient and averaging
  - Strongly convex vs. non-strongly convex

- **Going beyond stochastic gradient** (Le Roux, Schmidt, and Bach, 2012)
  - More than a single pass through the data
  - Linear (exponential) convergence rate for strongly convex functions

Supervised machine learning

- **Data**: n observations $(x_i, y_i) \in \mathcal{X} \times \mathcal{Y}$, $i = 1, \dots, n$, **i.i.d.**
- Prediction as a linear function $\theta^\top \Phi(x)$ of features $\Phi(x) \in \mathcal{F} = \mathbb{R}^p$
- **(regularized) empirical risk minimization**: find $\hat{\theta}$ solution of

$$\min_{\theta \in \mathcal{F}} \quad \frac{1}{n} \sum_{i=1}^{n} \ell\big(y_i, \theta^\top \Phi(x_i)\big) \quad + \quad \mu \Omega(\theta)$$

convex data fitting term $+$ regularizer

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- **Two fundamental questions**: (1) computing $\hat{\theta}$ and (2) analyzing $\hat{\theta}$

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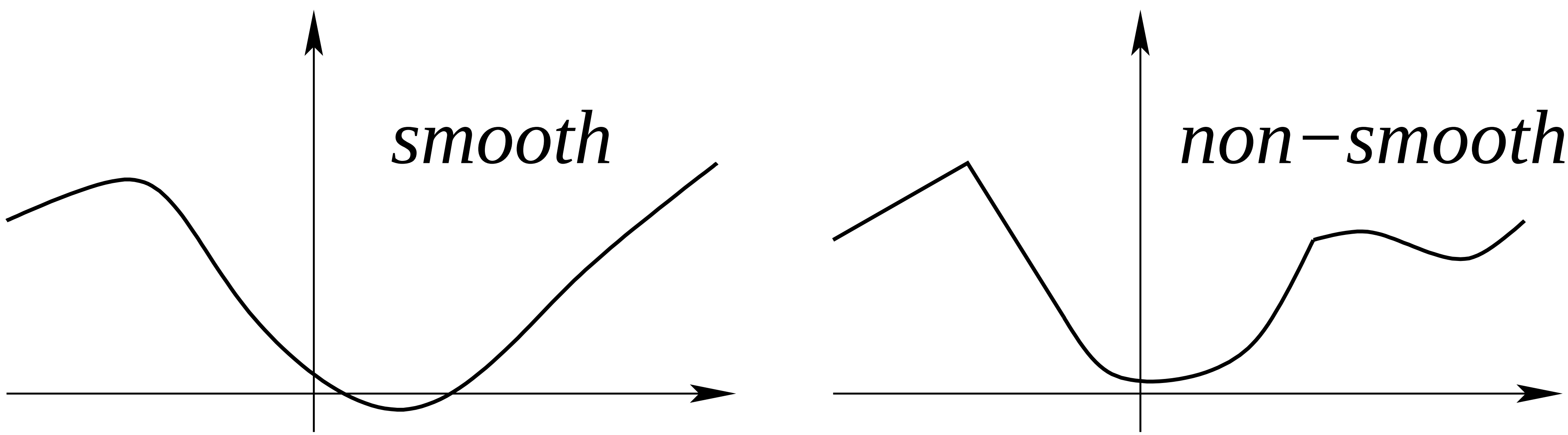
  - **May be tackled simultaneously**

Smoothness and strong convexity

- A function $g : \mathbb{R}^p \to \mathbb{R}$ is L-smooth if and only if it is differentiable and its gradient is L-Lipschitz-continuous

$$\forall \theta_1, \theta_2 \in \mathbb{R}^p, \ \|g'(\theta_1) - g'(\theta_2)\| \leqslant L\|\theta_1 - \theta_2\|$$

- If g is twice differentiable: $\forall \theta \in \mathbb{R}^p, \ g''(\theta) \preccurlyeq L \cdot Id$



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- **Machine learning**
  - with $g(\theta) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Hessian $\approx$ covariance matrix $\frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top$
  - Bounded data

Smoothness and strong convexity

- A function $g : \mathbb{R}^p \to \mathbb{R}$ is μ-strongly convex if and only if

$$\forall \theta_1, \theta_2 \in \mathbb{R}^p,\ \ g(\theta_1) \geqslant g(\theta_2) + \langle g'(\theta_2), \theta_1 - \theta_2 \rangle + \frac{\mu}{2}\|\theta_1 - \theta_2\|^2$$

- Equivalent definition: $\theta \mapsto g(\theta) - \frac{\mu}{2}\|\theta\|^2$ is convex

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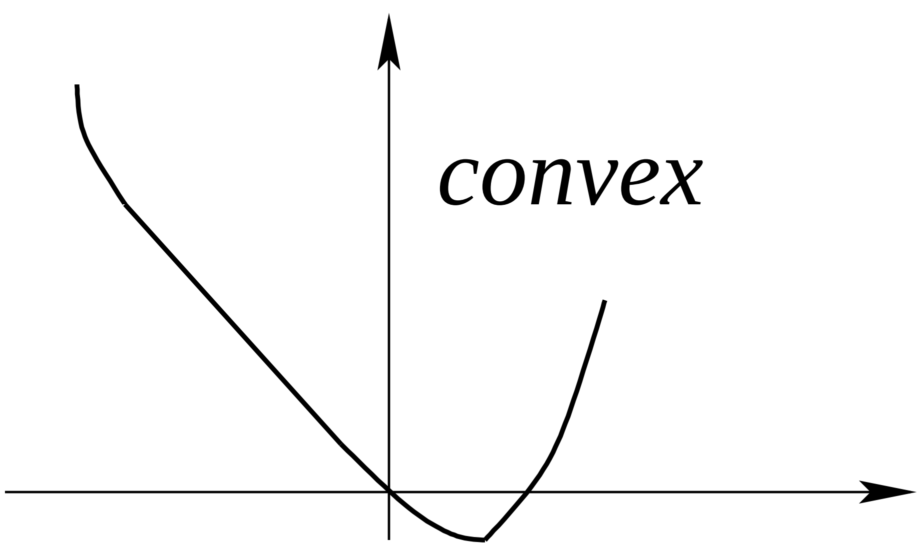

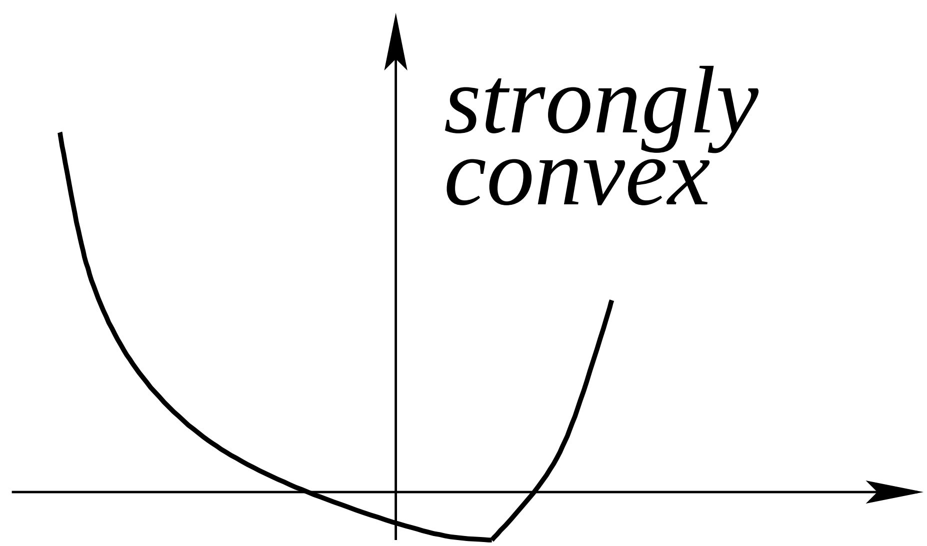

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  - Hessian $\approx$ covariance matrix $\frac{1}{n}\sum_{i=1}^n \Phi(x_i)\Phi(x_i)^\top$
  - Data with invertible covariance matrix (low correlation/dimension)
  - ... or with added regularization by $\frac{\mu}{2}\|\theta\|^2$

Stochastic approximation

- **Goal**: Minimizing a function f defined on a Hilbert space $\mathcal{H}$
  - given only unbiased estimates $f'_n(\theta_n)$ of its gradients $f'(\theta_n)$ at certain points $\theta_n \in \mathcal{H}$
- **Stochastic approximation**
  - Observation of $f'_n(\theta_n) = f'(\theta_n) + \varepsilon_n$, with ε_n = i.i.d. noise
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- **Machine learning - statistics**
  - **loss for a single pair of observations**: $\boxed{f_n(\theta) = \ell(y_n, \theta^\top \Phi(x_n))}$
  - $f(\theta) = \mathbb{E} f_n(\theta) = \mathbb{E}\,\ell(y_n, \theta^\top \Phi(x_n))$ = **generalization error**
  - Expected gradient: $f'(\theta) = \mathbb{E} f_n'(\theta) = \mathbb{E}\left\{\ell'(y_n, \theta^\top \Phi(x_n))\,\Phi(x_n)\right\}$

Convex smooth stochastic approximation

- **Key properties of f and/or f_n**
  - Smoothness: f_n L-smooth
  - Strong convexity: f μ-strongly convex

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- **Key algorithm:** Stochastic gradient descent (a.k.a. Robbins-Monro)

$$\boxed{\theta_n = \theta_{n-1} - \gamma_n f_n'(\theta_{n-1})}$$

  - Polyak-Ruppert averaging: $\bar{\theta}_n = \frac{1}{n}\sum_{k=0}^{n-1}\theta_k$
  - Which learning rate sequence γ_n? Classical setting: $\boxed{\gamma_n = Cn^{-\alpha}}$

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- **Desirable practical behavior**
  - Applicable (at least) to least-squares and logistic regression
  - Robustness to (potentially unknown) constants (L, μ)
  - Adaptivity to difficulty of the problem (e.g., strong convexity)

Convex stochastic approximation
Related work

- **Machine learning/optimization**
  - Known minimax rates of convergence (Nemirovski and Yudin, 1983; Agarwal et al., 2010)
    - Strongly convex: $O(n^{-1})$
    - Non-strongly convex: $O(n^{-1/2})$
  - Achieved with and/or without averaging (up to log terms)
  - Non-asymptotic analysis (high-probability bounds)
  - Online setting and regret bounds
  - Bottou and Le Cun (2005); Bottou and Bousquet (2008); Hazan et al. (2007); Shalev-Shwartz and Srebro (2008); Shalev-Shwartz et al. (2007, 2009); Xiao (2010); Duchi and Singer (2009)
  - Nesterov and Vial (2008); Nemirovski et al. (2009)

Convex stochastic approximation
Related work

- **Stochastic approximation**
  - Asymptotic analysis
  - Non convex case with strong convexity around the optimum
  - $\gamma_n = Cn^{-\alpha}$ with $\alpha = 1$ is not robust to the choice of C
  - $\alpha \in (1/2, 1)$ is robust with averaging
  - Broadie et al. (2009); Kushner and Yin (2003); Kul′chitskiĭ and Mozgovoĭ (1991); Fabian (1968)
  - Polyak and Juditsky (1992); Ruppert (1988)

Problem set-up - General assumptions

- **Unbiased gradient estimates**:
  - $f_n(\theta)$ is of the form $h(z_n, \theta)$, where z_n is an i.i.d. sequence
  - e.g., $f_n(\theta) = h(z_n, \theta) = \ell(y_n, \theta^\top \Phi(x_n))$ with $z_n = (x_n, y_n)$
  - NB: can be generalized

- **Variance of estimates**: There exists $\sigma^2 \geqslant 0$ such that for all $n \geqslant 1$, $\mathbb{E}(\|f_n'(\theta^*) - f'(\theta^*)\|^2) \leqslant \sigma^2$, where θ^* is a global minimizer of f

Problem set-up - Smoothness/convexity assumptions

- **Smoothness of f_n**: For each $n \geqslant 1$, the function f_n is a.s. convex, differentiable with L-Lipschitz-continuous gradient f_n':
  - Bounded data

Problem set-up - Smoothness/convexity assumptions

- **Smoothness of f_n**: For each $n \geqslant 1$, the function f_n is a.s. convex, differentiable with L-Lipschitz-continuous gradient f_n':
  - Bounded data

- **Strong convexity of f**: The function f is strongly convex with respect to the norm $\|\cdot\|$, with convexity constant $\mu > 0$:
  - Invertible population covariance matrix
  - or regularization by $\frac{\mu}{2}\|\theta\|^2$

Summary of new results (Bach and Moulines, 2011)

- Stochastic gradient descent with learning rate $\gamma_n = Cn^{-\alpha}$
- **Strongly convex smooth objective functions**
  - Old: $O(n^{-1})$ rate achieved without averaging for $\alpha = 1$
  - New: $O(n^{-1})$ rate achieved with averaging for $\alpha \in [1/2, 1]$
  - Non-asymptotic analysis with explicit constants
  - Forgetting of initial conditions
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- **Proof technique**
  - Derive deterministic recursion for $\delta_n = \mathbb{E}\|\theta_n - \theta^*\|^2$

$$\delta_n \leqslant (1 - 2\mu\gamma_n + 2L^2\gamma_n^2)\delta_{n-1} + 2\sigma^2\gamma_n^2$$

  - Mimic SA proof techniques in a non-asymptotic way

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  - Non-asymptotic analysis with explicit constants
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  - Robustness to the choice of C
- **Convergence rates** for $\mathbb{E}\|\theta_n - \theta^*\|^2$ and $\mathbb{E}\|\bar{\theta}_n - \theta^*\|^2$
  - no averaging: $O\Big(\dfrac{\sigma^2\gamma_n}{\mu}\Big) + O(e^{-\mu n \gamma_n})\|\theta_0 - \theta^*\|^2$
  - averaging: $\dfrac{\operatorname{tr} H(\theta^*)^{-1}}{n} + \mu^{-1}O(n^{-2\alpha} + n^{-2+\alpha}) + O\Big(\dfrac{\|\theta_0 - \theta^*\|^2}{\mu^2 n^2}\Big)$

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- **Non-strongly convex smooth objective functions**
  - Old: $O(n^{-1/2})$ rate achieved with averaging for $\alpha = 1/2$
  - New: $O(\max\{n^{1/2-3\alpha/2}, n^{-\alpha/2}, n^{\alpha-1}\})$ rate achieved without averaging for $\alpha \in [1/3, 1]$
- **Take-home message**
  - Use $\alpha = 1/2$ with averaging to be adaptive to strong convexity

Conclusions / Extensions
Stochastic approximation for machine learning

- **Mixing convex optimization and statistics**
  - Non-asymptotic analysis through moment computations
  - Averaging with longer steps is (more) robust and adaptive

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- **Future/current work - open problems**
  - High-probability through all moments $\mathbb{E}\|\theta_n - \theta^*\|^{2d}$
  - Analysis for logistic regression using self-concordance (Bach, 2010)
  - Including a non-differentiable term (Xiao, 2010; Lan, 2010)
  - Non-random errors (Schmidt, Le Roux, and Bach, 2011)
  - Line search for stochastic gradient
  - Non-parametric stochastic approximation
  - Online estimation of uncertainty
  - Going beyond a single pass through the data

Going beyond a single pass over the data

- **Stochastic approximation**
  - Assumes infinite data stream
  - Observations are used only once
  - Directly minimizes testing cost $\mathbb{E}_z h(\theta, z) = \mathbb{E}_{(x,y)} \ell(y, \theta^\top \Phi(x))$

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- **Machine learning practice**
  - Finite data set $(z_1, \dots, z_n)$
  - Multiple passes
  - Minimizes training cost $\frac{1}{n}\sum_{i=1}^n h(\theta, z_i) = \frac{1}{n}\sum_{i=1}^n \ell(y_i, \theta^\top \Phi(x_i))$
  - Need to regularize (e.g., by the ℓ_2-norm) to avoid overfitting

Stochastic vs. deterministic methods

- Minimizing $g(\theta) = \frac{1}{n}\sum_{i=1}^{n} f_i(\theta)$ with $f_i(\theta) = \ell\big(y_i, \theta^\top \Phi(x_i)\big) + \mu\Omega(\theta)$

- Batch gradient descent: $\theta_t = \theta_{t-1} - \gamma_t g'(\theta_{t-1}) = \theta_{t-1} - \frac{\gamma_t}{n}\sum_{i=1}^{n} f_i'(\theta_{t-1})$

  - Linear (e.g., exponential) convergence rate
  - Iteration complexity is linear in n

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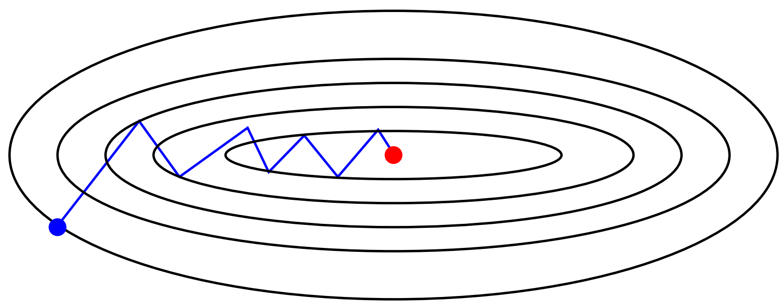

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- Stochastic gradient descent: $\theta_t = \theta_{t-1} - \gamma_t f'_{i(t)}(\theta_{t-1})$
  - Sampling with replacement: $i(t)$ random element of $\{1, \dots, n\}$
  - Convergence rate in $O(1/t)$
  - Iteration complexity is independent of n

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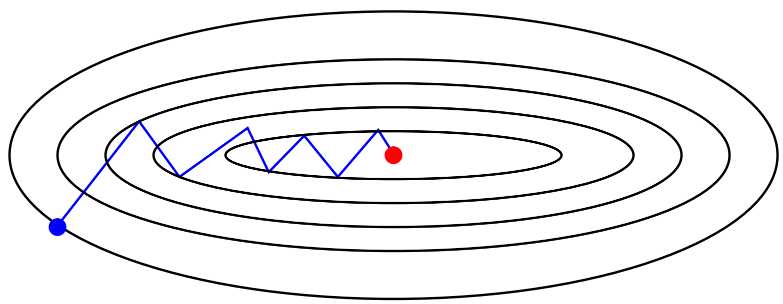

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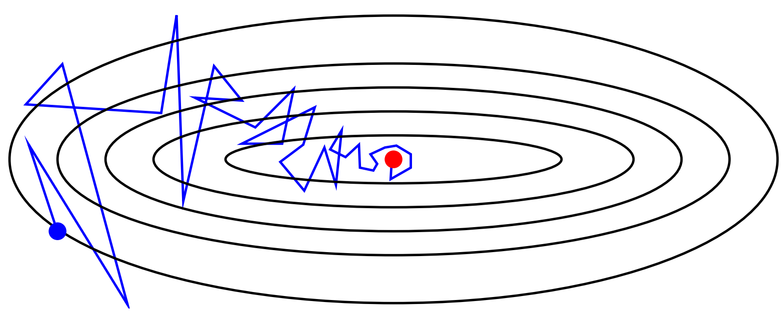

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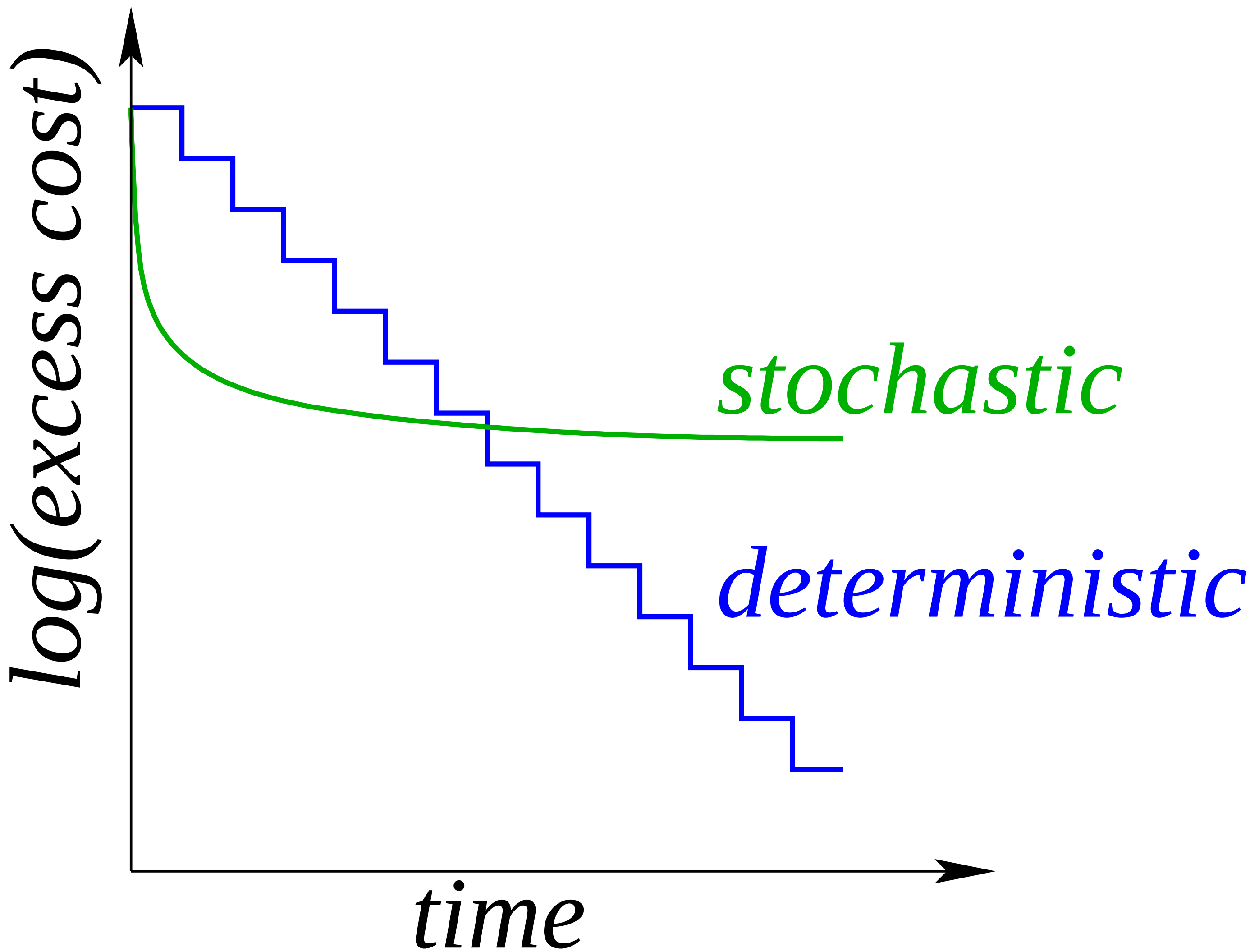

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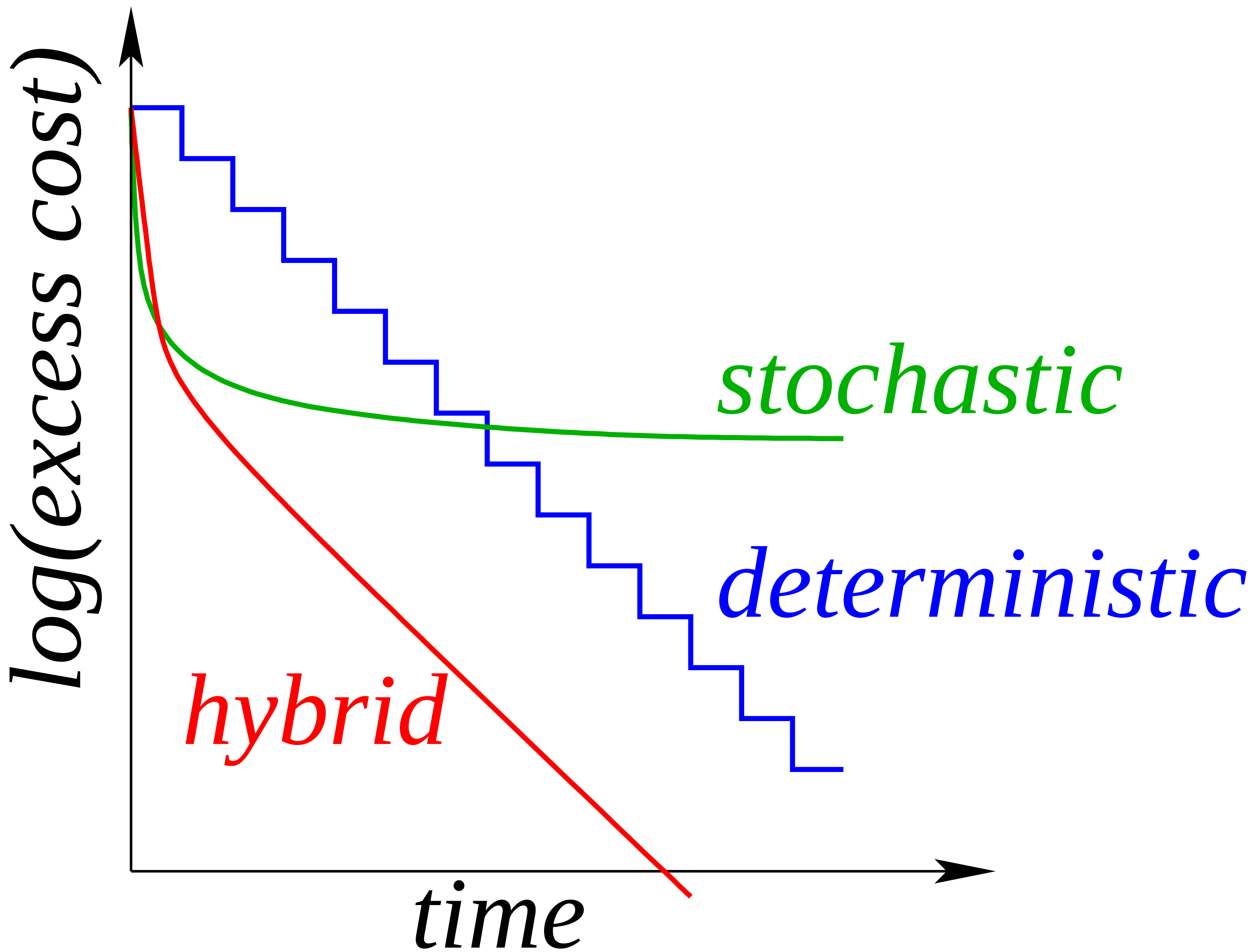

Accelerating gradient methods - Related work

- **Nesterov acceleration**
  - Nesterov (1983, 2004)
  - Better linear rate but still $O(n)$ iteration cost

- **Hybrid methods, incremental average gradient, increasing batch size**
  - Bertsekas (1997); Blatt et al. (2008); Friedlander and Schmidt (2011)
  - Linear rate, but iterations make full passes through the data.

Accelerating gradient methods - Related work

- **Momentum, gradient/iterate averaging, stochastic version of accelerated batch gradient methods**
  - Polyak and Juditsky (1992); Tseng (1998); Sunehag et al. (2009); Ghadimi and Lan (2010); Xiao (2010)
  - Can improve constants, but still have sublinear $O(1/t)$ rate

- **Constant step-size stochastic gradient (SG), accelerated SG**
  - Kesten (1958); Delyon and Juditsky (1993); Solodov (1998); Nedic and Bertsekas (2000)
  - Linear convergence, but only up to a fixed tolerance.

- **Stochastic methods in the dual**
  - Shalev-Shwartz and Zhang (2012)
  - Linear rate but limited choice for the f_i's

Stochastic average gradient (Le Roux, Schmidt, and Bach, 2012)

- **Stochastic average gradient** (SAG) iteration
  - Keep in memory the gradients of all functions f_i, $i = 1, \dots, n$
  - Random selection $i(t) \in \{1, \dots, n\}$ with replacement
  - Iteration: $\theta_t = \theta_{t-1} - \frac{\gamma_t}{n} \sum_{i=1}^n y_i^t$ with $y_i^t = \begin{cases} f_i'(\theta_{t-1}) & \text{if } i = i(t) \\ y_i^{t-1} & \text{otherwise} \end{cases}$

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- Stochastic version of incremental average gradient (Blatt et al., 2008)

- Extra memory requirement
  - Supervised machine learning
    - If $f_i(\theta) = \ell_i(y_i, \Phi(x_i)^\top \theta)$, then $f_i'(\theta) = \ell_i'(y_i, \Phi(x_i)^\top \theta)\, \Phi(x_i)$
    - Only need to store n real numbers

Stochastic average gradient
Convergence analysis - I

- Assume each f_i is L-smooth and $\hat{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$ is μ-strongly convex

- **Constant step size** $\gamma_t = \frac{1}{2nL}$:

$$\mathbb{E}\big[\|\theta_t - \theta^*\|^2\big] \leqslant \left(1 - \frac{\mu}{8Ln}\right)^t \Big[3\|\theta_0 - \theta^*\|^2 + \frac{9\sigma^2}{4L^2}\Big]$$

  – Linear rate with iteration cost independent of n ...
  – ... but, same behavior as batch gradient and IAG (cyclic version)

- **Proof technique**
  – Designing a quadratic Lyapunov function for a n-th order non-linear stochastic dynamical system

Stochastic average gradient
Convergence analysis - II

- Assume each f_i is L-smooth and $\hat{f} = \frac{1}{n}\sum_{i=1}^{n} f_i$ is μ-strongly convex

- **Constant step size** $\gamma_t = \frac{1}{2n\mu}$, if $\frac{\mu}{L} \geqslant \frac{8}{n}$

$$\mathbb{E}\big[\hat{f}(\theta_t) - \hat{f}(\theta^*)\big] \leqslant C\left(1 - \frac{1}{8n}\right)^t$$

with $C = \left[\frac{16L}{3n}\|\theta_0 - \theta^*\|^2 + \frac{4\sigma^2}{3n\mu}\left(8\log\left(1 + \frac{\mu n}{4L}\right) + 1\right)\right]$

  - Linear rate with iteration cost independent of n
  - Linear convergence rate "independent" of the condition number
  - After each pass through the data, constant error reduction

Rate of convergence comparison

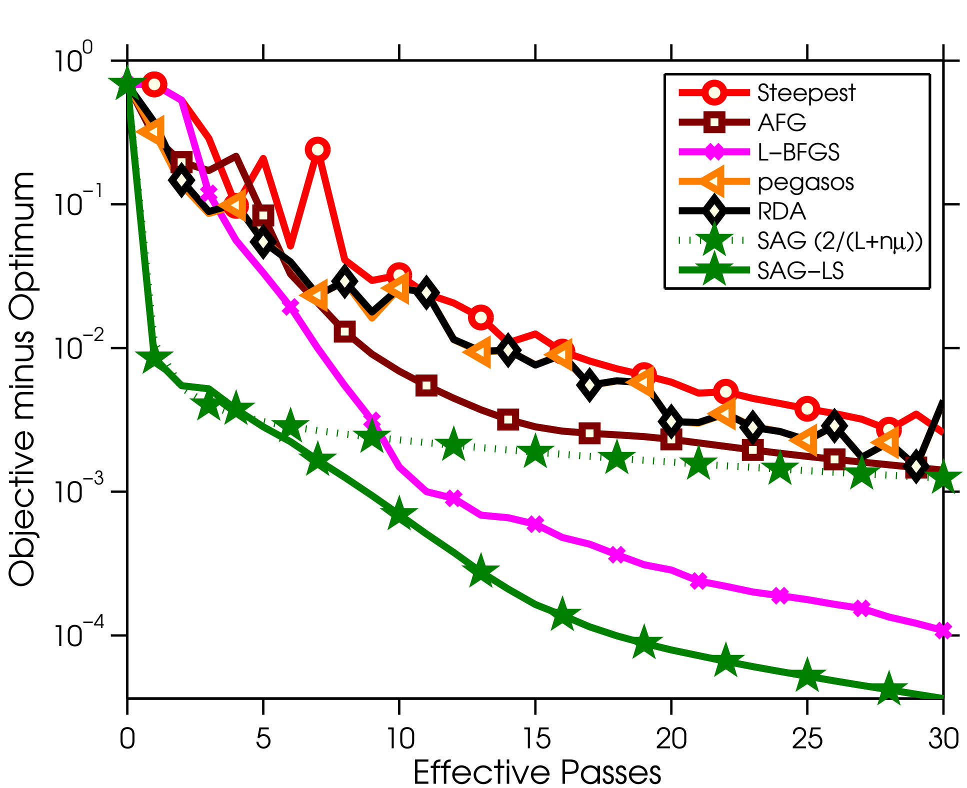
- Assume that $L = 100$, $\mu = .01$, and $n = 80000$
  - Full gradient method has rate
  $$\left(1 - \frac{\mu}{L}\right) = 0.9999$$
  - Accelerated gradient method has rate
  $$\left(1 - \sqrt{\frac{\mu}{L}}\right) = 0.9900$$
  - Running n iterations of SAG for the same cost has rate
  $$\left(1 - \frac{1}{8n}\right)^n = 0.8825$$
  - *Fastest possible* first-order method has rate
  $$\left(\frac{\sqrt{L}-\sqrt{\mu}}{\sqrt{L}+\sqrt{\mu}}\right)^2 = 0.9608$$
- **Beating two lower bounds** (with additional assumptions)
  - (1) stochastic gradient and (2) full gradient

Stochastic average gradient
Implementation details and extensions

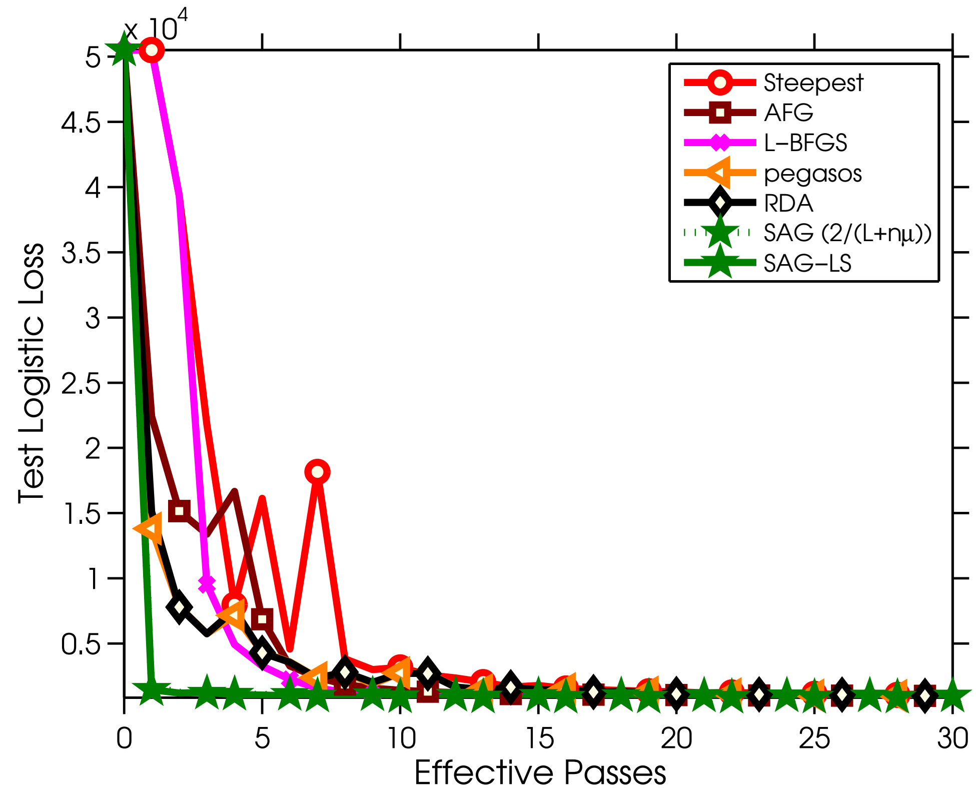
- The algorithm can use sparsity in the features to reduce the storage and iteration cost
- Grouping functions together can further reduce the memory requirement
- We have obtained good performance when L is not known with a heuristic line-search
- Algorithm allows non-uniform sampling
- Possibility of making proximal, coordinate-wise, and Newton-like variants

Stochastic average gradient
Simulation experiments

- `protein` dataset ($n = 145751$, $p = 74$)
- Dataset split in two (training/testing)



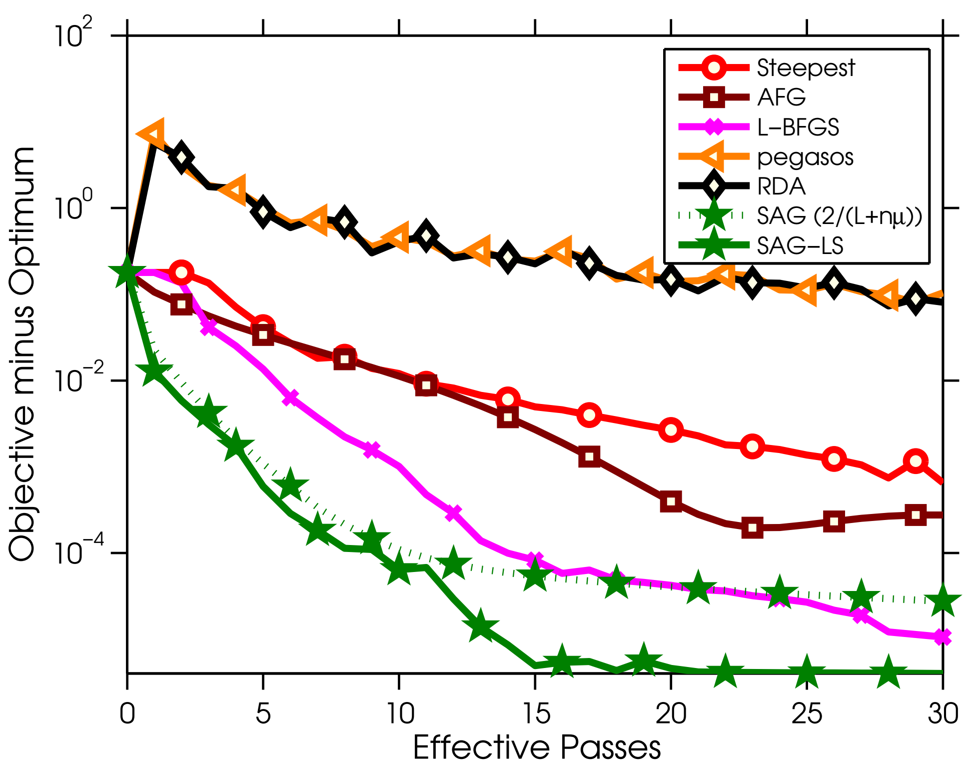
Training cost



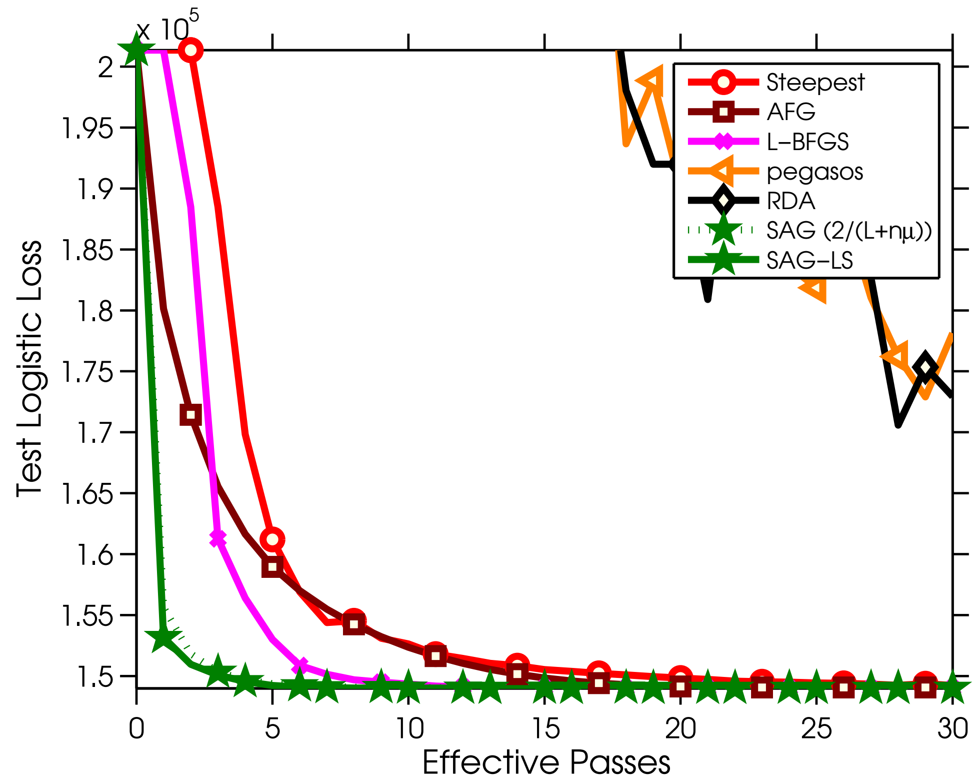
Testing cost

Stochastic average gradient
Simulation experiments

- cover type dataset ($n = 581012$, $p = 54$)
- Dataset split in two (training/testing)



Training cost



Testing cost

Conclusions / Extensions
Stochastic average gradient

- **Going beyond a single pass through the data**
  - Keep memory of all gradients for finite training sets
  - Linear convergence rate with $O(1)$ iteration complexity
  - Randomization leads to easier analysis and faster rates
  - Beyond machine learning

Conclusions / Extensions
Stochastic average gradient

- **Going beyond a single pass through the data**
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- **Future/current work - open problems**
  - Including a non-differentiable term
  - Line search
  - Using second-order information or non-uniform sampling
  - Going beyond finite training sets (bound on testing cost)
  - Non strongly-convex case

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