Graphical Models
Discrete Inference and Learning
Lecture 1

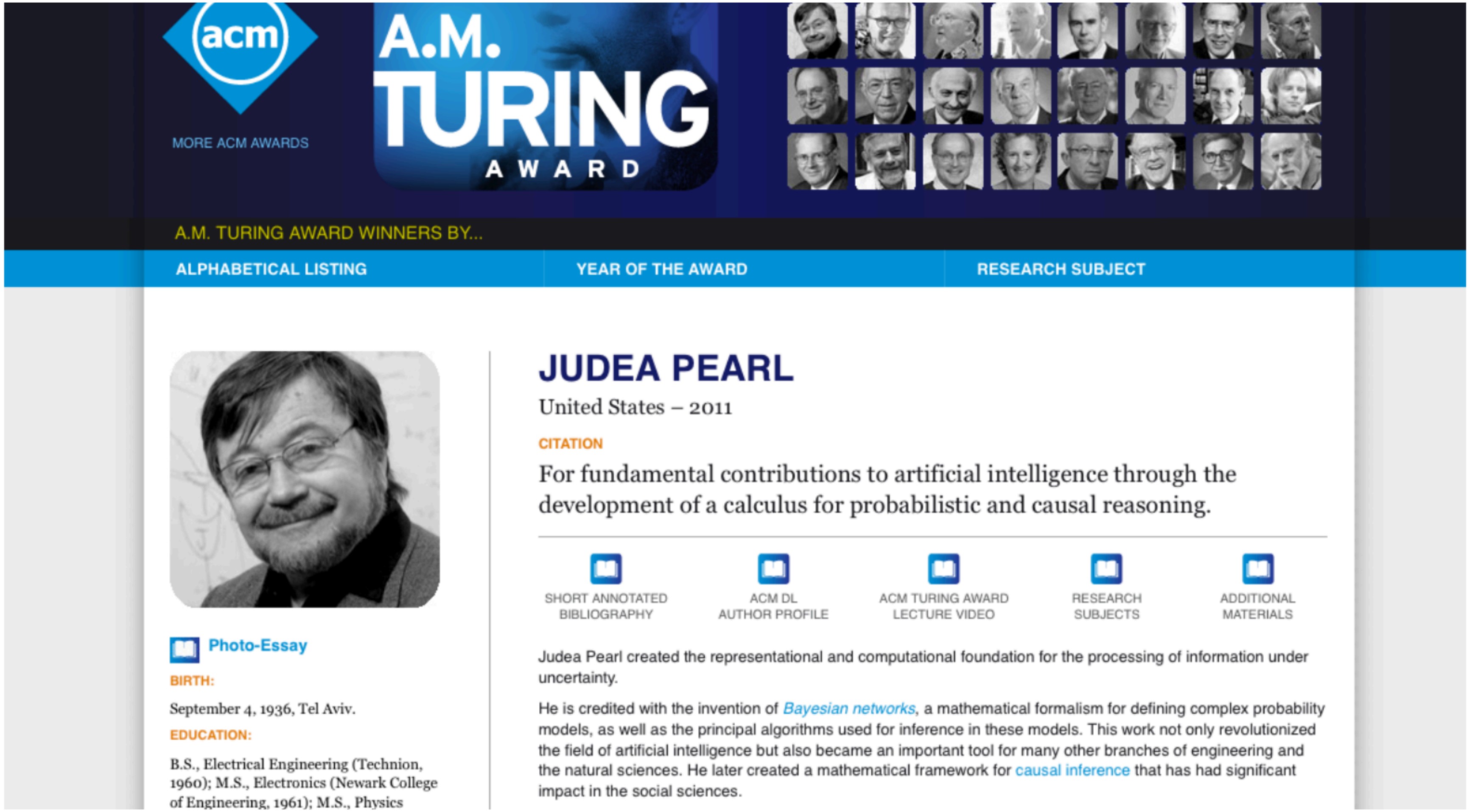
MVA

2019 – 2020

http://thoth.inrialpes.fr/~alahari/disinflearn

Slides based on material from Stephen Gould, Pushmeet Kohli, Nikos Komodakis, M. Pawan Kumar, Carsten Rother, Daphne Koller, Dhruv Batra

Graphical Models ?



Slide courtesy: Dhruv Batra

What this class is about?

- Making **global** predictions from **local** observations

Inference

- Learning such models from large quantities of data

Learning

Motivation

- Consider the example of medical diagnosis

Predisposing factors
Symptoms
Test results

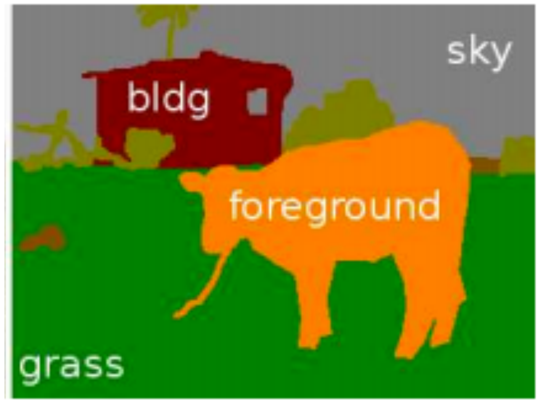
↓

Diseases
Treatment outcomes

Slide inspired by PGM course, Daphne Koller

Motivation

- A very different example: image segmentation



e.g., [He et al., 2004; Shotton et al., 2006; Gould et al., 2009]

Millions of pixels
Colours / features

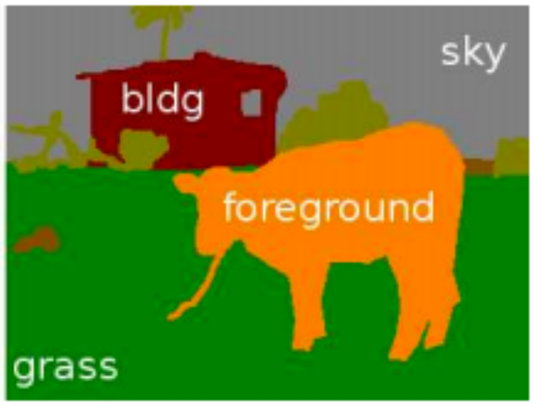
↓

Pixel labels
{building, grass, cow, sky}

Slide inspired by PGM course, Daphne Koller

Motivation

- What do these two problems have in common?



Slide inspired by PGM course, Daphne Koller

Motivation

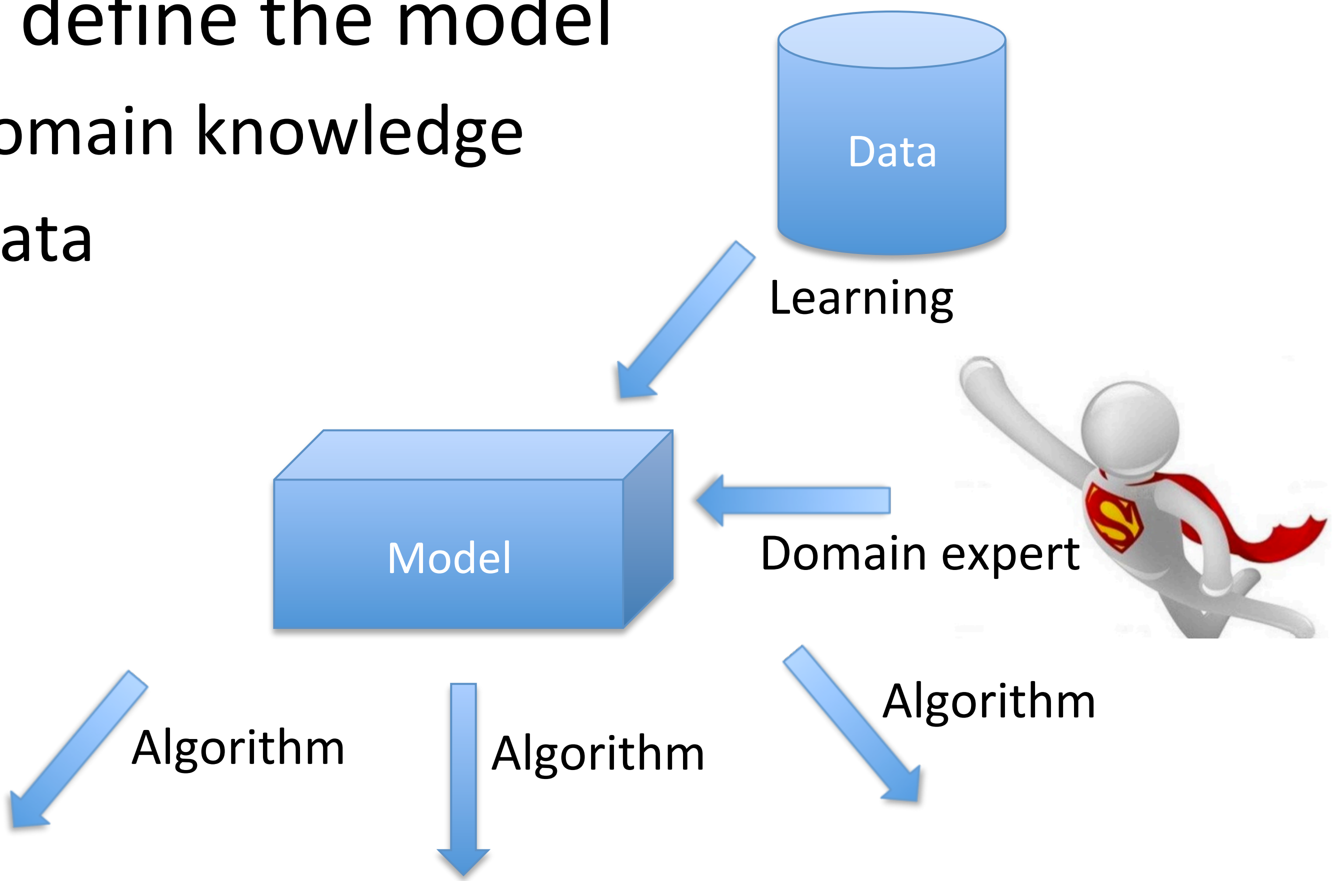
- What do these two problems have in common?
  - Many variables
  - Uncertainty about the correct answer

Graphical Models (or Probabilistic Graphical Models) provide a framework to address these problems

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- First, it is a model: a declarative representation
- Can also define the model
  - with domain knowledge
  - from data



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(Probabilistic) Graphical Models

- Why probabilistic ?
- To model uncertainty
- Uncertainty due to:
  - Partial knowledge of state of the world
  - Noisy observations
  - Phenomena not observed by the model
  - Inherent stochasticity

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- Probability theory provides
  - Standalone representation with clear semantics
  - Reasoning patterns (conditioning, decision making)
  - Learning methods

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- Why graphical ?
- Intersection of ideas from probability theory and computer science
  - To represent large number of variables

Predisposing factors
Symptoms
Test results

Millions of pixels
Colours / features

Random variables $\mathbf{Y_1,\ldots,Y_n}$

Goal: capture uncertainty through joint distribution $P(Y_1,\ldots,Y_n)$

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

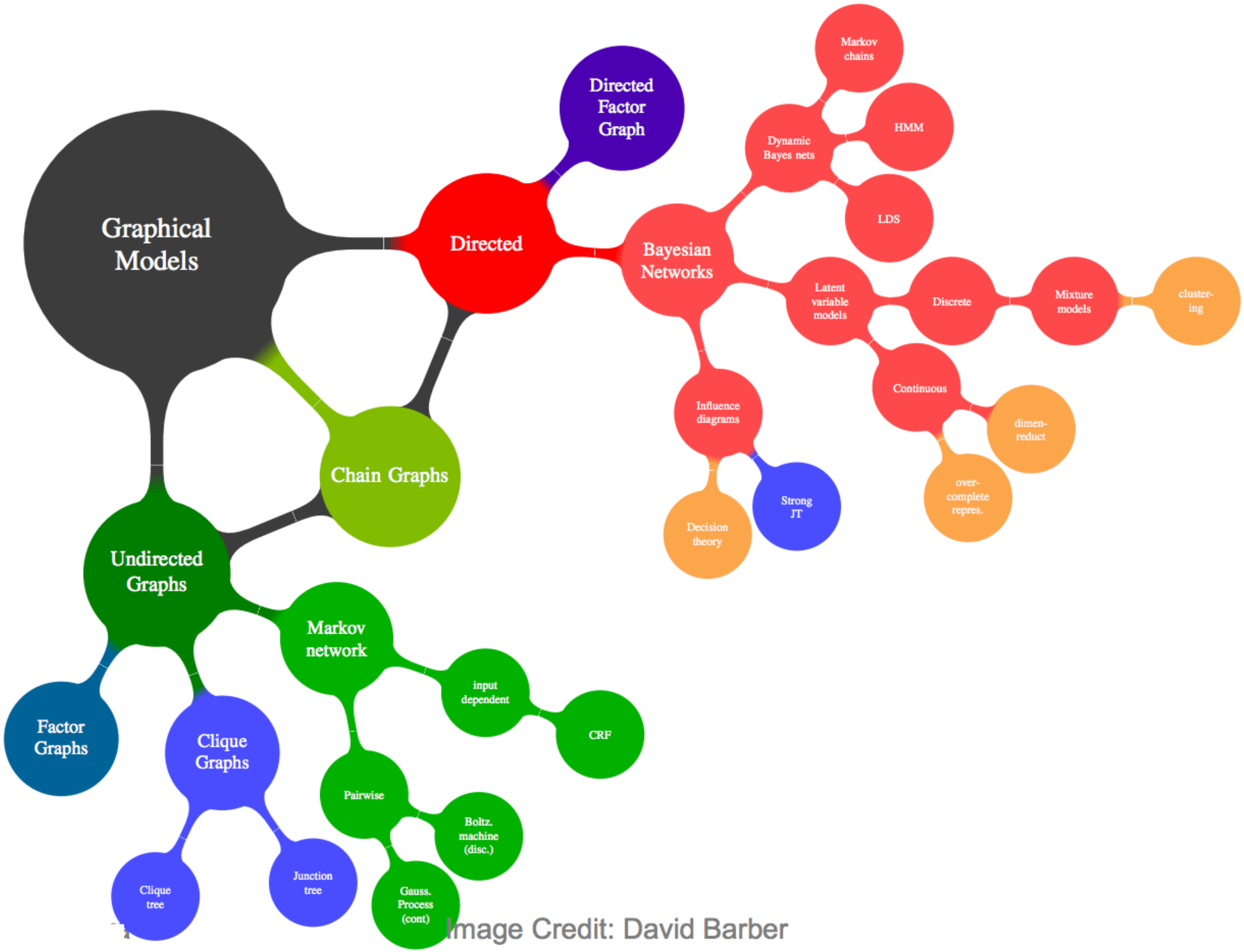
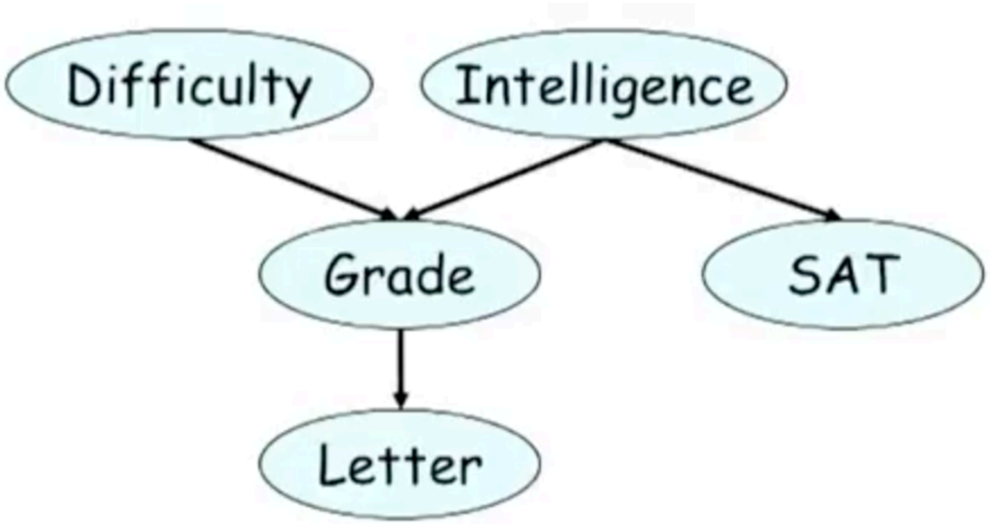


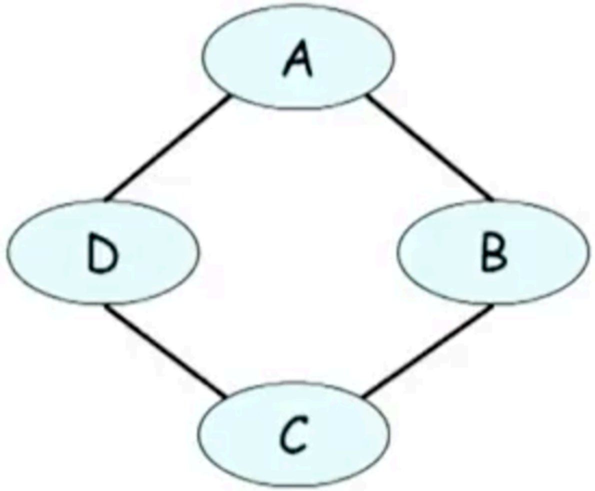
Image Credit: David Barber

(Probabilistic) Graphical Model

- Examples



Bayesian network
(directed graph)

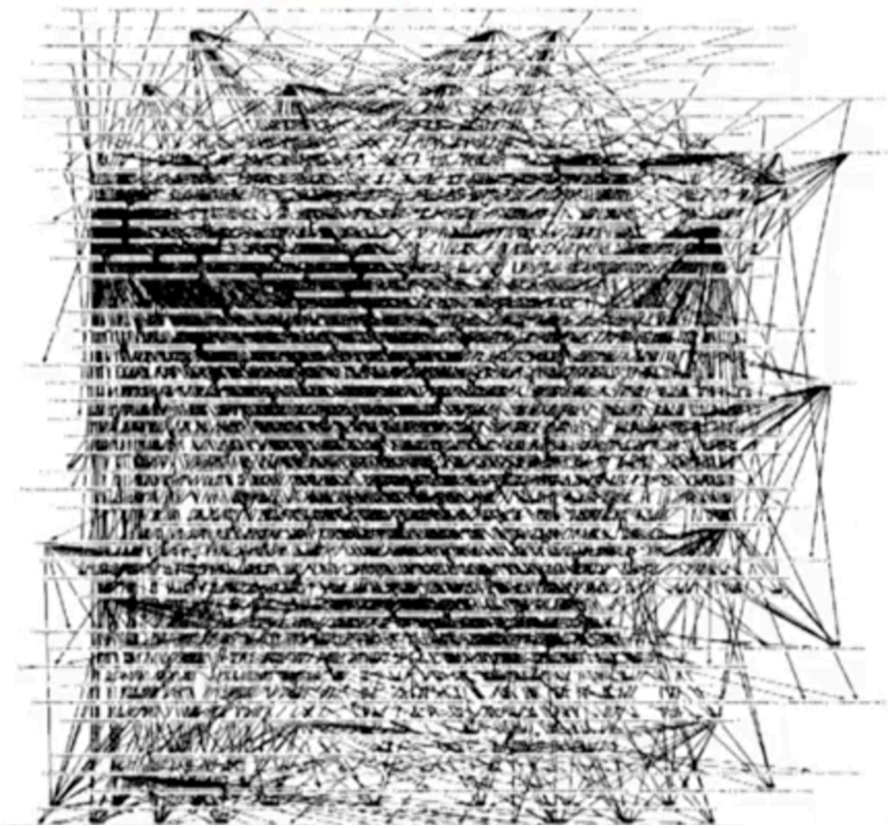


Markov network
(undirected graph)

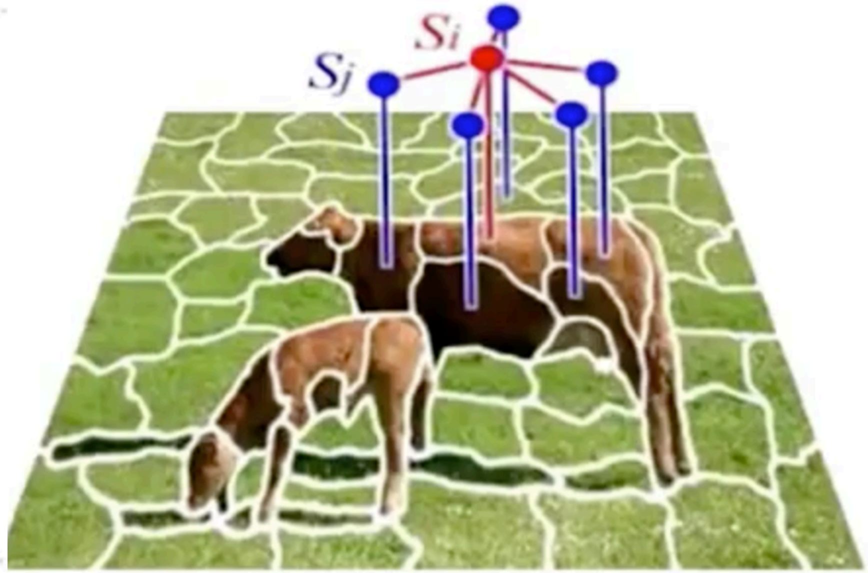
Figure courtesy: D. Koller

(Probabilistic) Graphical Model

- Examples



Diagnosis network: Pradhan et al., UAI'94



Segmentation network (Courtesy D. Koller)

(Probabilistic) Graphical Model

- Intuitive & compact data structure

- Efficient reasoning through general-purpose algorithms

- Sparse parameterization
  - Through expert knowledge, or
  - Learning from data

Slide inspired by PGM course, Daphne Koller

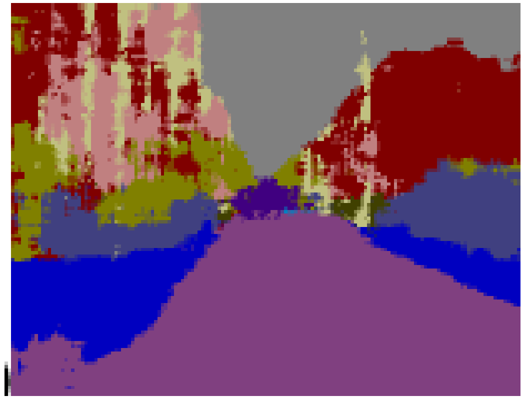
(Probabilistic) Graphical Model

- Many many applications
  - Medical diagnosis
  - Fault diagnosis
  - Natural language processing
  - Traffic analysis
  - Social network models
  - Message decoding
  - Computer vision: segmentation, 3D, pose estimation
  - Speech recognition
  - Robot localization & mapping

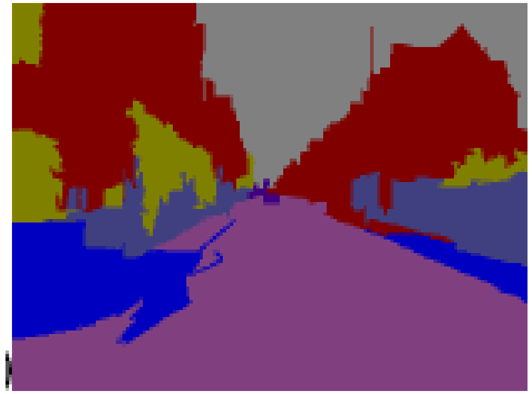
Slide courtesy: PGM course, Daphne Koller

Image segmentation

Image



No graphical model

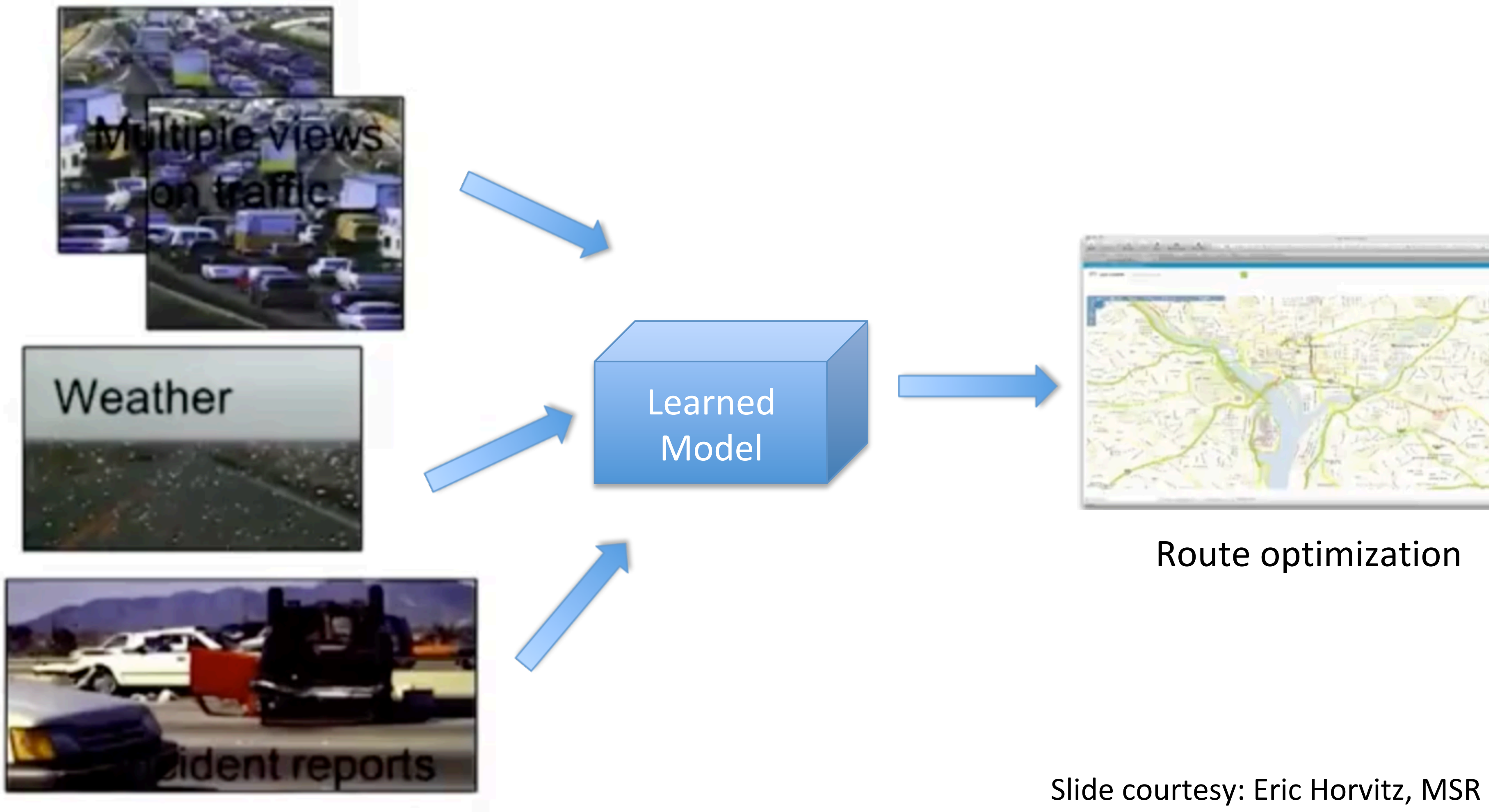


With graphical model

Sturgess et al., 2009

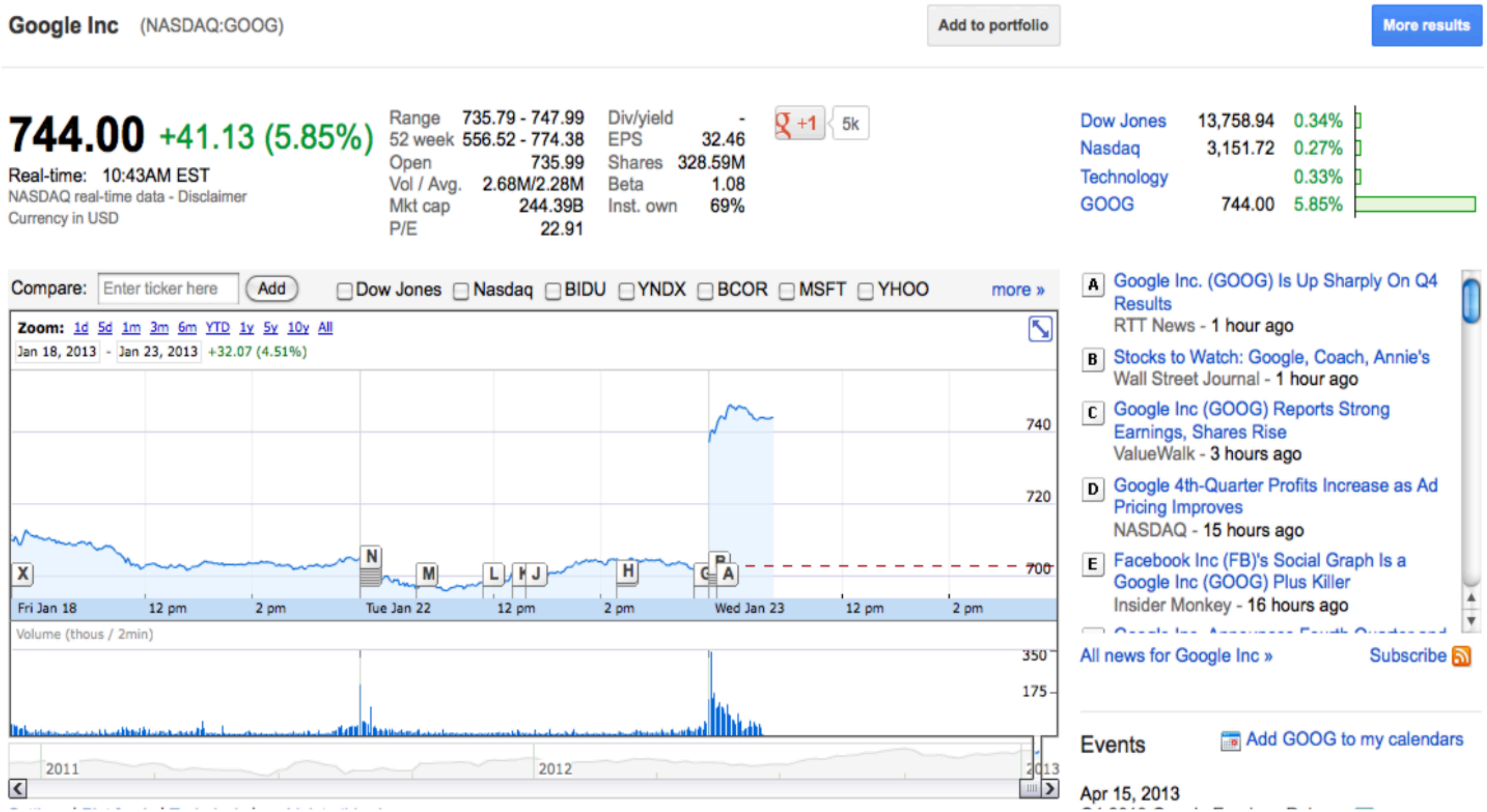
Multi-sensor integration: Traffic

- Learn from historical data to make predictions



Slide courtesy: Eric Horvitz, MSR

Stock market



Slide courtesy: Dhruv Batra

Going global: Local ambiguity

- Text recognition

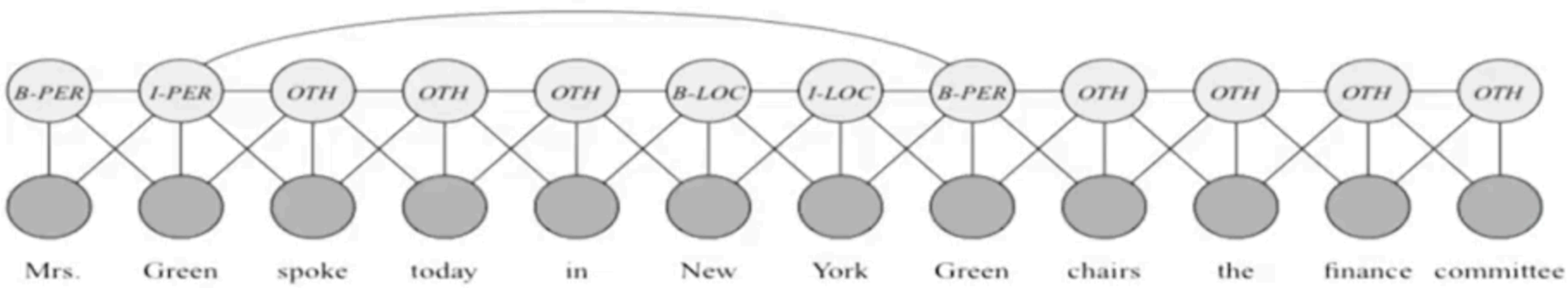
Smyth et al., 1994

Slide courtesy: Dhruv Batra

Going global: Local ambiguity

- Textual information extraction

e.g., Mrs. Green spoke today in New York. Green chairs the financial committee.



Slide courtesy: PGM course, Daphne Koller

Overview of the course

- Representation
  - How do we store $P(Y_1,\ldots Y_n)$
  - Directed and undirected (model implications/assumptions)
- Inference
  - Answer questions with the model
  - Exact and approximate (marginal/most probable estimate)
- Learning
  - What model is right for data
  - Parameters and structure

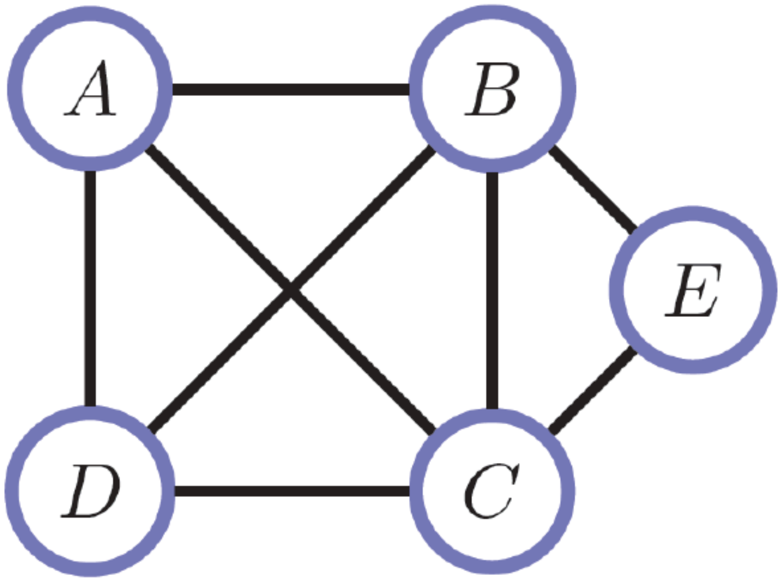
Slide inspired by D. Batra, D. Koller 's courses

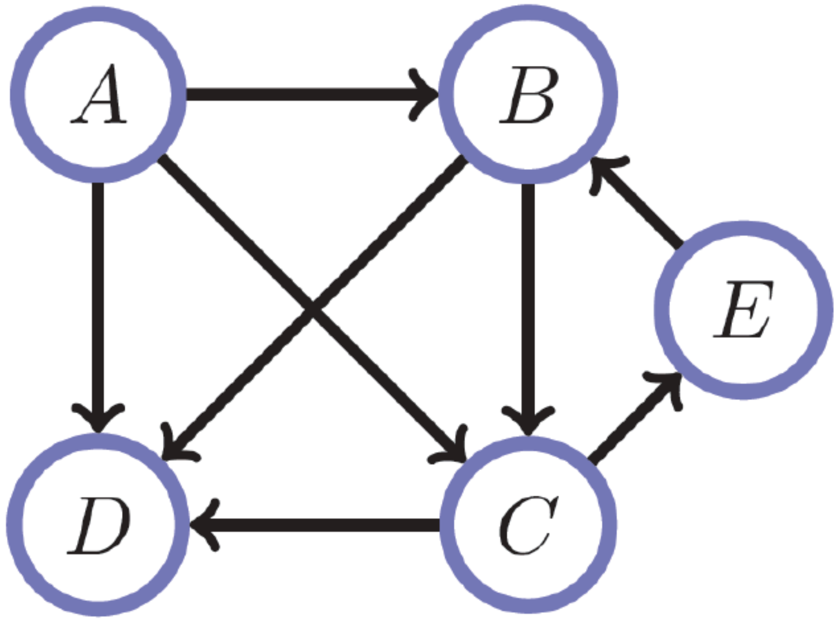
First, a recap of basics

Graphs

- Concepts
  - Definition of G
  - Vertices/Nodes
  - Edges
  - Directed vs Undirected
  - Neighbours vs Parent/Child
  - Degree vs In/Out degree
  - Walk vs Path vs Cycle

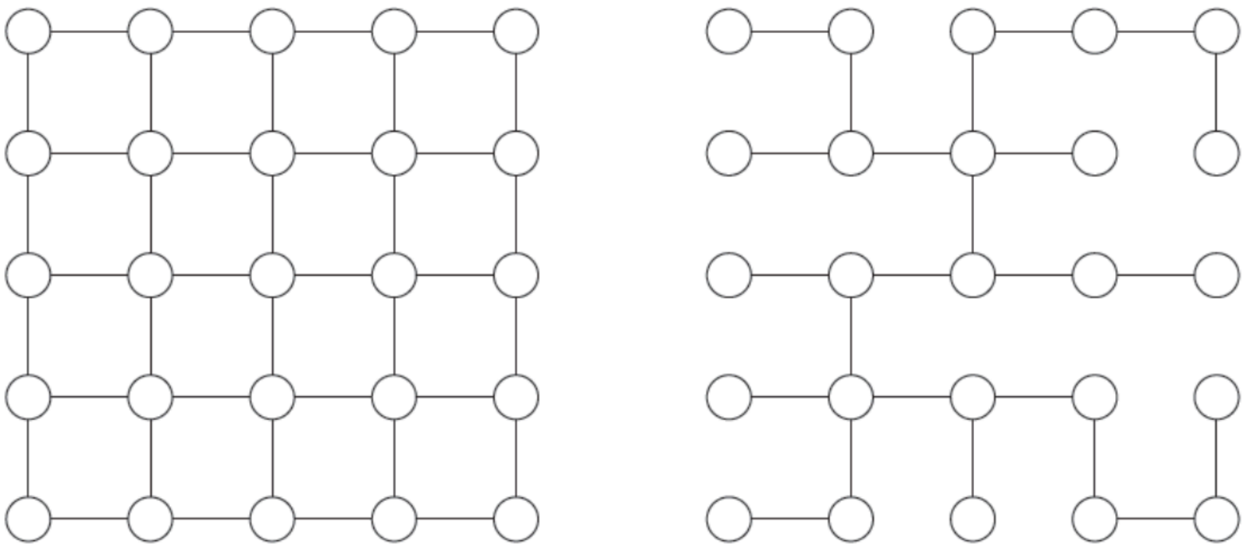
Graphs





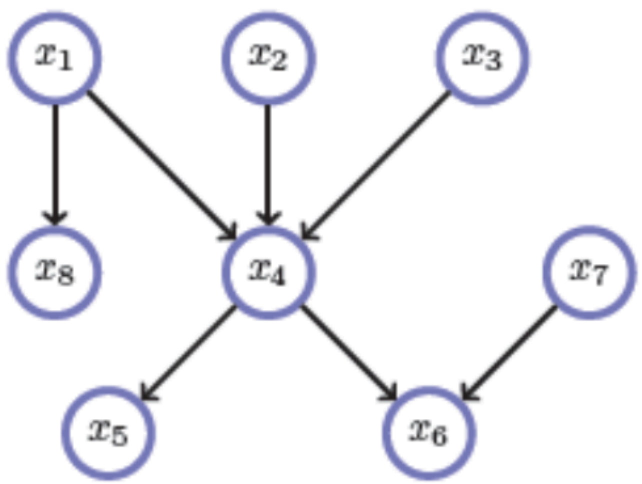
Special graphs

- Trees: undirected graph, no cycles
- Spanning tree: Same set of vertices, but subset of edges, connected and no cycles



Slide courtesy: D. Batra

Directed acyclic graphs (DAGs)



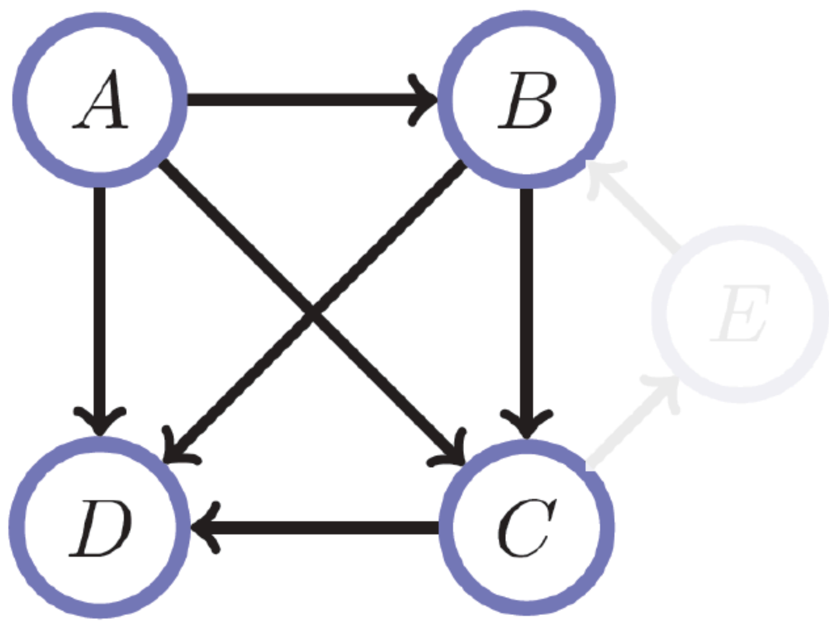


Figure courtesy: D. Batra

Interpreting Probability

- What does P(A) mean?
- Frequentist view
  - Limit $N \rightarrow \infty$, #(A is true)/N
  - i.e., limiting frequency of a repeating non-deterministic event
- Bayesian view
  - P(A) is your belief about A

Slide courtesy: D. Batra

Joint distribution

- 3 variables
  - Intelligence (I)
  - Difficulty (D)
  - Grade (G)

- Independent parameters?

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^0	g^2	0.168
i^0	d^0	g^3	0.126
i^0	d^1	g^1	0.009
i^0	d^1	g^2	0.045
i^0	d^1	g^3	0.126
i^1	d^0	g^1	0.252
i^1	d^0	g^2	0.0224
i^1	d^0	g^3	0.0056
i^1	d^1	g^1	0.06
i^1	d^1	g^2	0.036
i^1	d^1	g^3	0.024

Example courtesy: PGM course, Daphne Koller

Conditioning

- Condition on g^1

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^0	g^2	0.168
i^0	d^0	g^3	0.126
i^0	d^1	g^1	0.009
i^0	d^1	g^2	0.045
i^0	d^1	g^3	0.126
i^1	d^0	g^1	0.252
i^1	d^0	g^2	0.0224
i^1	d^0	g^3	0.0056
i^1	d^1	g^1	0.06
i^1	d^1	g^2	0.036
i^1	d^1	g^3	0.024

Example courtesy: PGM course, Daphne Koller

Conditioning

- $P(Y = y \mid X = x)$
- Informally,
  - What do you believe about Y=y when I tell you X=x ?

- P(France wins Euro 2020) ?
- What if I tell you:
  - France won the world cup 2018
  - Hasn't had catastrophic results since ☺

Slide based on D. Batra's course

Conditioning: Reduction

- Condition on g^1

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.06

Example courtesy: PGM course, Daphne Koller

Conditioning: Renormalization

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.06

$P(I, D, g^1)$

Unnormalized measure

I	D	Prob.
i^0	d^0	0.282
i^0	d^1	0.02
i^1	d^0	0.564
i^1	d^1	0.134

$P(I, D \mid g^1)$

Example courtesy: PGM course, Daphne Koller

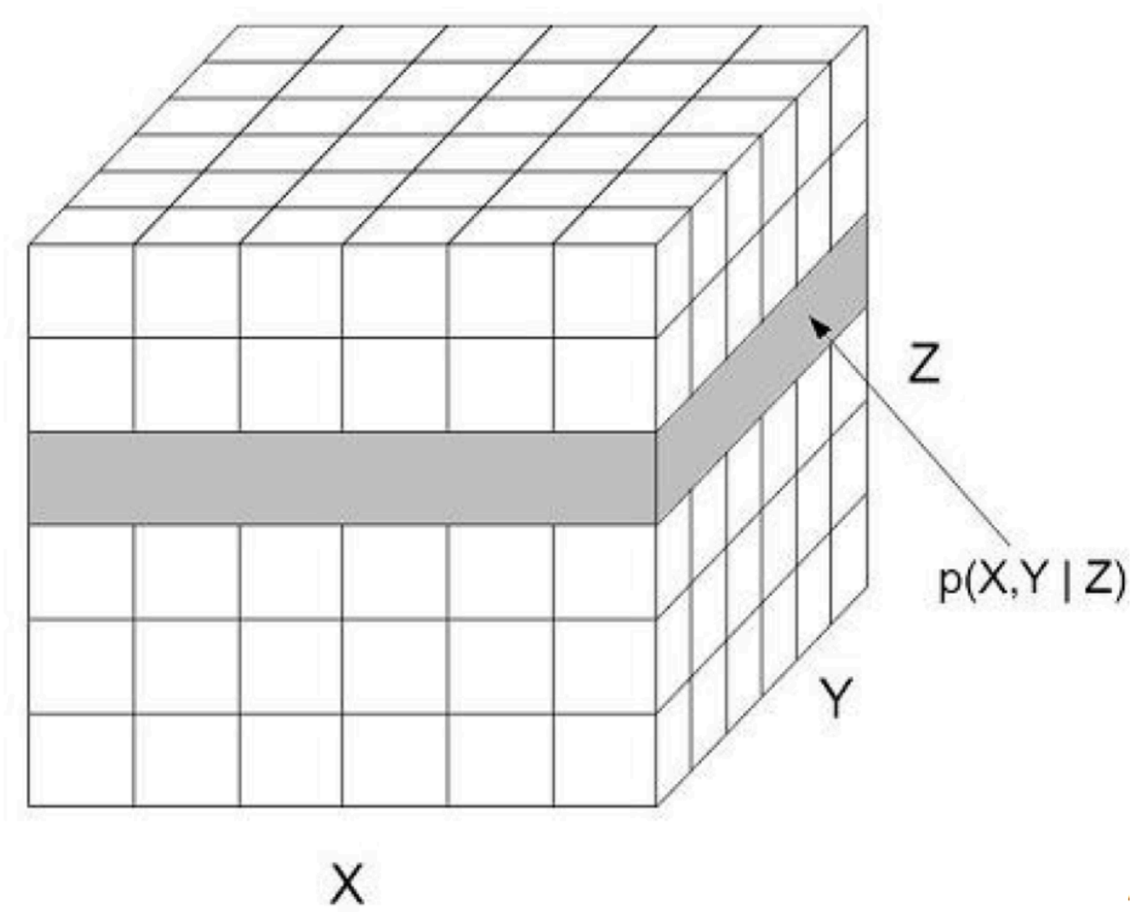
Conditional probability distribution

- Example $P(G \mid I, D)$

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

Example courtesy: PGM course, Daphne Koller

Conditional probability distribution



$$p(x, y \mid Z = z) = \frac{p(x, y, z)}{p(z)}$$

Slide courtesy: Erik Sudderth

Marginalization

P(I,D)

I	D	Prob.
i^0	d^0	0.282
i^0	d^1	0.02
i^1	d^0	0.564
i^1	d^1	0.134

Marginalize I

D	Prob.
d^0	0.846
d^1	0.154

Example courtesy: PGM course, Daphne Koller

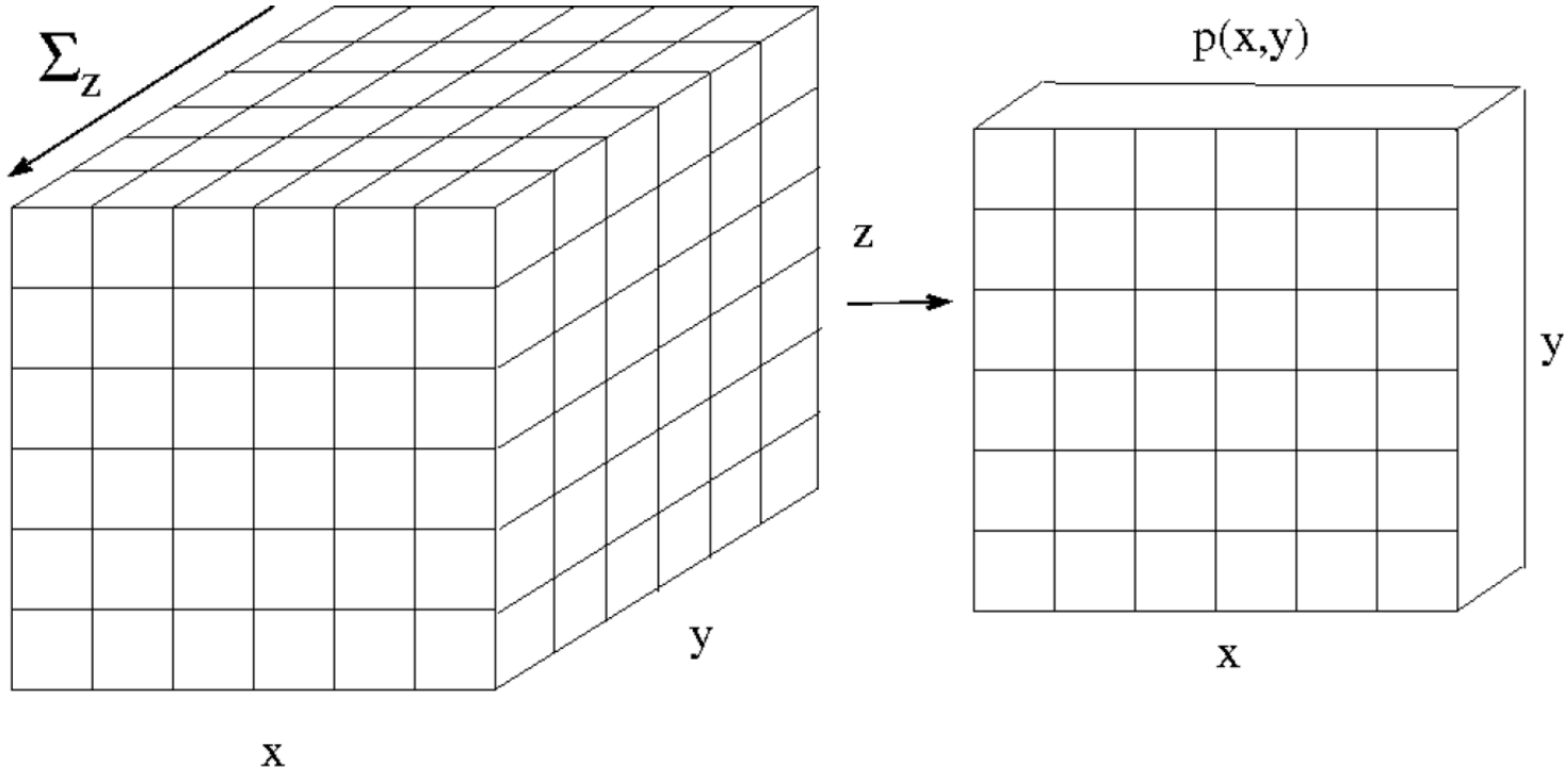
Marginalization

- Events
  - P(A) = P(A and B) + P(A and not B)

- Random variables
  - $P(X = x) = \sum_{y} P(X = x, Y = y)$

Slide courtesy: Dhruv Batra

Marginalization



$$p(x, y) = \sum_{z \in \mathcal{Z}} p(x, y, z)$$

$$p(x) = \sum_{y \in \mathcal{Y}} p(x, y)$$

Slide courtesy: Erik Sudderth

Factors

- A factor $\Phi(Y_1,\ldots,Y_k)$

  Φ: $Val(Y_1,\ldots,Y_k) \rightarrow R$

- Scope = $\{Y_1,\ldots,Y_k\}$

Factors

- Example: $P(D, I, G)$

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^0	g^2	0.168
i^0	d^0	g^3	0.126
i^0	d^1	g^1	0.009
i^0	d^1	g^2	0.045
i^0	d^1	g^3	0.126
i^1	d^0	g^1	0.252
i^1	d^0	g^2	0.0224
i^1	d^0	g^3	0.0056
i^1	d^1	g^1	0.06
i^1	d^1	g^2	0.036
i^1	d^1	g^3	0.024

Example courtesy: PGM course, Daphne Koller

Factors

- Example: $P(D, I, g^1)$

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.06

What is the scope here?

Example courtesy: PGM course, Daphne Koller

General factors

- Not necessarily for probabilities

A	B	ϕ
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

Example courtesy: PGM course, Daphne Koller

Factor product

a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.1
a^2	b^2	0
a^3	b^1	0.3
a^3	b^2	0.9

b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

a^1	b^1	c^1	$0.5 \cdot 0.5 = 0.25$
a^1	b^1	c^2	$0.5 \cdot 0.7 = 0.35$
a^1	b^2	c^1	$0.8 \cdot 0.1 = 0.08$
a^1	b^2	c^2	$0.8 \cdot 0.2 = 0.16$
a^2	b^1	c^1	$0.1 \cdot 0.5 = 0.05$
a^2	b^1	c^2	$0.1 \cdot 0.7 = 0.07$
a^2	b^2	c^1	$0 \cdot 0.1 = 0$
a^2	b^2	c^2	$0 \cdot 0.2 = 0$
a^3	b^1	c^1	$0.3 \cdot 0.5 = 0.15$
a^3	b^1	c^2	$0.3 \cdot 0.7 = 0.21$
a^3	b^2	c^1	$0.9 \cdot 0.1 = 0.09$
a^3	b^2	c^2	$0.9 \cdot 0.2 = 0.18$

Example courtesy: PGM course, Daphne Koller

Factor marginalization

a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.05
a^2	b^1	c^2	0.07
a^2	b^2	c^1	0
a^2	b^2	c^2	0
a^3	b^1	c^1	0.15
a^3	b^1	c^2	0.21
a^3	b^2	c^1	0.09
a^3	b^2	c^2	0.18

a^1	c^1	0.33
a^1	c^2	0.51
a^2	c^1	0.05
a^2	c^2	0.07
a^3	c^1	0.24
a^3	c^2	0.39

Example courtesy: PGM course, Daphne Koller

Factor reduction

a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.05
a^2	b^1	c^2	0.07
a^2	b^2	c^1	0
a^2	b^2	c^2	0
a^3	b^1	c^1	0.15
a^3	b^1	c^2	0.21
a^3	b^2	c^1	0.09
a^3	b^2	c^2	0.18

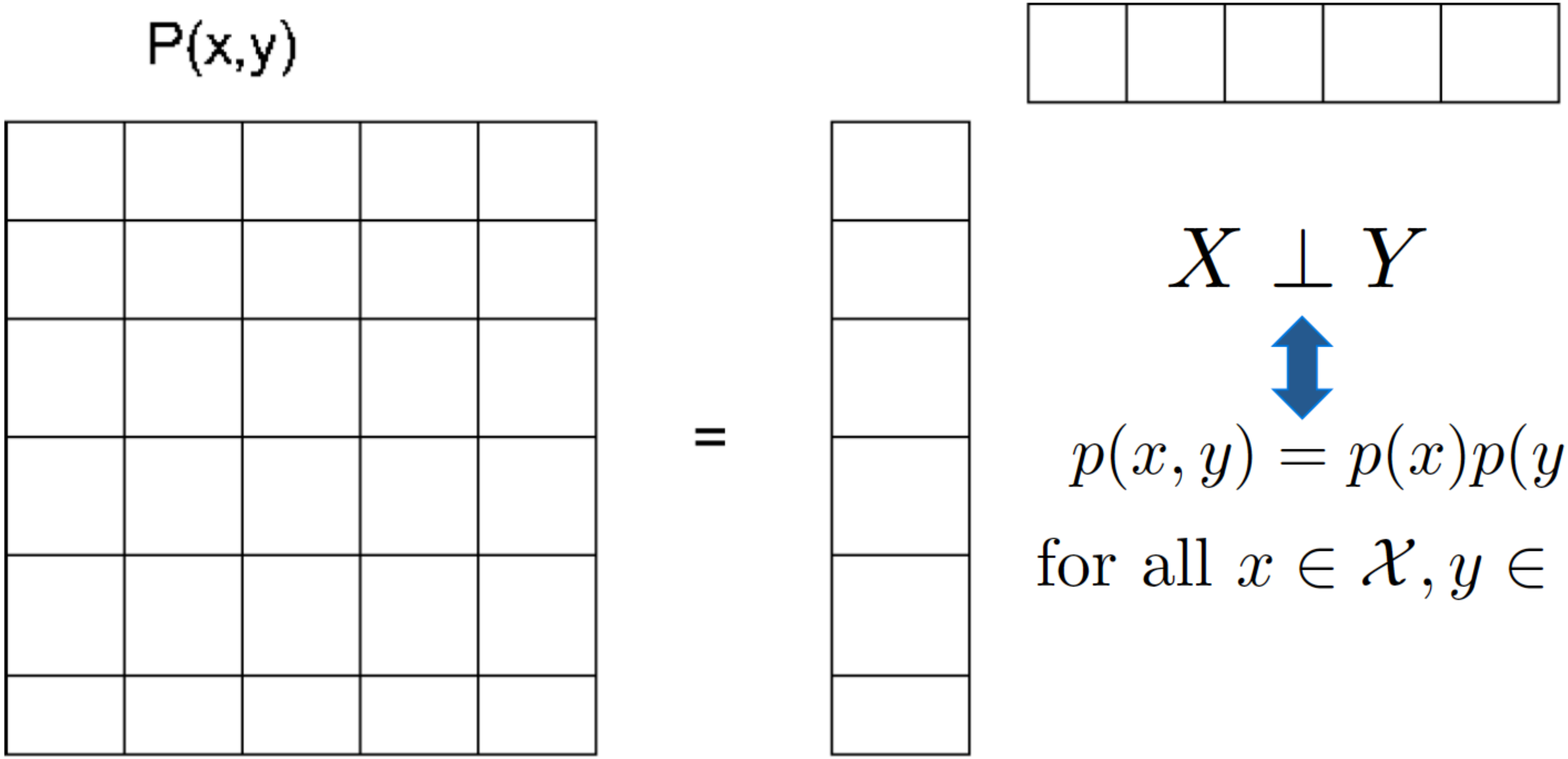
a^1	b^1	c^1	0.25
a^1	b^2	c^1	0.08
a^2	b^1	c^1	0.05
a^2	b^2	c^1	0
a^3	b^1	c^1	0.15
a^3	b^2	c^1	0.09

Why factors ?

- Building blocks for defining distributions in high-dimensional spaces

- Set of basic operations for manipulating these distributions

Independent random variables



$$X \perp Y$$

⇕

$$p(x, y) = p(x)p(y)$$

$$\text{for all } x \in \mathcal{X}, y \in \mathcal{Y}$$

Slide courtesy: Erik Sudderth

Marginal independence

- **Sets** of variables **X**, **Y**

- **X** is independent of **Y**
  - Shorthand: $P \vdash (\mathbf{X} \perp \mathbf{Y})$

- **Proposition:** *P* satisfies $(\mathbf{X} \perp \mathbf{Y})$ if and only if
  - $P(\mathbf{X}=\mathbf{x}, \mathbf{Y}=\mathbf{y}) = P(\mathbf{X}=\mathbf{x})\, P(\mathbf{Y}=\mathbf{y})$, $\forall \mathbf{x} \in Val(\mathbf{X}), \mathbf{y} \in Val(\mathbf{Y})$

Slide courtesy: Dhruv Batra

Conditional independence

- **Sets** of variables **X**, **Y**, **Z**
- **X** is independent of **Y** given **Z** if
  - Shorthand: $P \vdash (\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$
  - For $P \vdash (\mathbf{X} \perp \mathbf{Y} \mid \emptyset)$, write $P \vdash (\mathbf{X} \perp \mathbf{Y})$
- **Proposition:** P satisfies $(\mathbf{X} \perp \mathbf{Y} \mid \mathbf{Z})$ if and only if
  - $P(\mathbf{X},\mathbf{Y}|\mathbf{Z}) = P(\mathbf{X}|\mathbf{Z})\, P(\mathbf{Y}|\mathbf{Z})$, $\quad \forall \mathbf{x} \in Val(\mathbf{X}), \mathbf{y} \in Val(\mathbf{Y}), \mathbf{z} \in Val(\mathbf{Z})$

Slide courtesy: Dhruv Batra

Bayes Rule

- Simple yet profound
- Concepts
  - Likelihood
    - How much does a certain hypothesis explain the data?
  - Prior
    - What do you believe before seeing any data?
  - Posterior
    - What do we believe after seeing the data?

Slide courtesy: Dhruv Batra

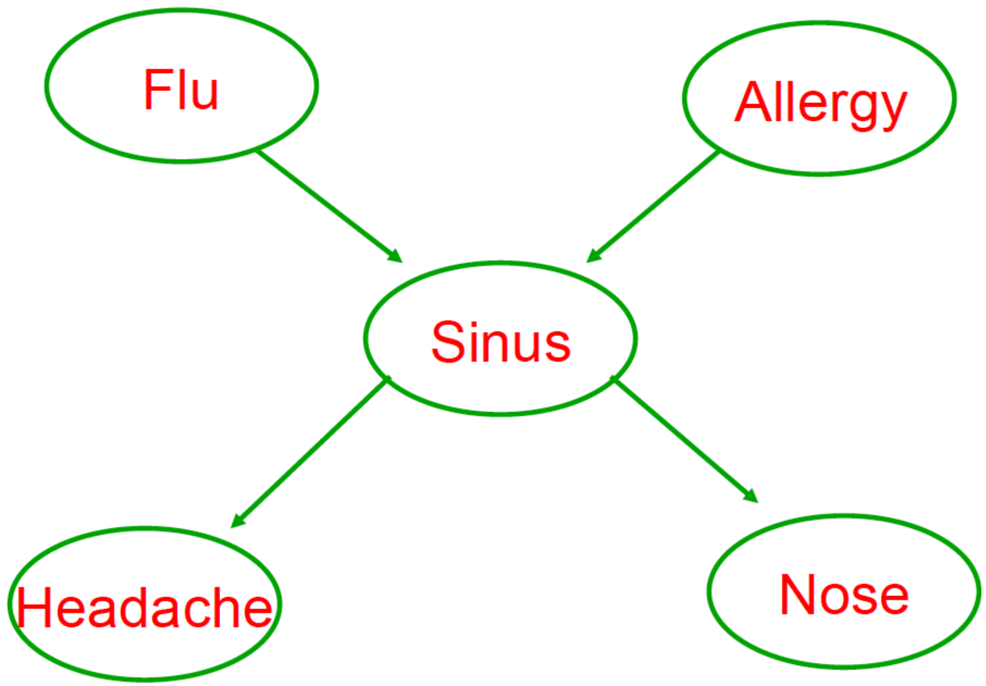
Bayesian Networks

- DAGs
  - nodes represent variables in the Bayesian sense
  - edges represent conditional dependencies
- Example
  - Suppose that we know the following:
    - The flu causes sinus inflammation
    - Allergies cause sinus inflammation
    - Sinus inflammation causes a runny nose
    - Sinus inflammation causes headaches
  - How are these connected ?

Slide courtesy: Dhruv Batra

Bayesian Networks

- Example



Slide courtesy: Dhruv Batra

Bayesian Networks

- A general Bayes net
  - Set of random variables
  - DAG: encodes independence assumptions
  - Conditional probability trees
  - Joint distribution

$$P(Y_1, \ldots, Y_n) = \prod_{i=1}^{n} P(Y_i \mid \mathrm{Pa}_{Y_i})$$

Slide courtesy: Dhruv Batra

Bayesian Networks

- A general Bayes net
  - How many parameters ?
    - Discrete variables $Y_1,\ldots,Y_n$
    - Graph: Defines parents of Y_i, i.e., (Pa_{Y_i})
    - CPTs: $P(Y_i|Pa_{Y_i})$

Slide courtesy: Dhruv Batra

Markov nets

- Set of random variables
- Undirected graph
  - Encodes independence assumptions
- Factors

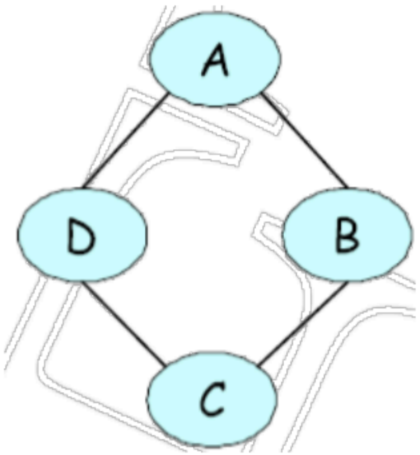
Comparison to Bayesian Nets ?

Slide courtesy: Dhruv Batra

Pairwise MRFs

- Composed of pairwise factors
  - A function of two variables
  - Can also have unary terms

- Example



$\phi_1[A,B]$		
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

$\phi_2[B,C]$		
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

$\phi_3[C,D]$		
c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

$\phi_4[D,A]$		
d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

Slide courtesy: Dhruv Batra

Markov Nets: Computing probabilities

- Can only compute ratio of probabilities directly

$\phi_1[A,B]$			$\phi_2[B,C]$			$\phi_3[C,D]$			$\phi_4[D,A]$		
a^0	b^0	30	b^0	c^0	100	c^0	d^0	1	d^0	a^0	100
a^0	b^1	5	b^0	c^1	1	c^0	d^1	100	d^0	a^1	1
a^1	b^0	1	b^1	c^0	1	c^1	d^0	100	d^1	a^0	1
a^1	b^1	10	b^1	c^1	100	c^1	d^1	1	d^1	a^1	100

- Need to normalize with a **partition function**
  - Hard ! (sum over all possible assignments)
- In Bayesian Nets, can do by multiplying CPTs

Slide courtesy: Dhruv Batra

Markov nets ←→ Factorization

- Given an undirected graph H over variables $Y=\{Y_1,\ldots,Y_n\}$
- A distribution P factorizes over H if there exist
  - Subsets of variables $S^i \subseteq Y$ s.t. S^i are fully-connected in H
  - Non-negative potentials (factors) $\Phi_1(S^1),\ldots,\Phi_m(S^m)$: clique potentials
  - Such that

$$P(Y_1,\ldots,Y_n) = \frac{1}{Z}\prod_{i=1}^{m}\Phi_i(S^i)$$

Slide courtesy: Dhruv Batra

Conditional Markov Random Fields

- Also known as: Markov networks, undirected graphical models, MRFs
- Note: Not making a distinction between CRFs and MRFs
- $\mathbf{X} \in \mathcal{X}$: observed random variables
- $\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathcal{Y}$: output random variables
- $\mathbf{Y}_c$ are subset of variables for clique $c \subseteq \{1, \ldots, n\}$
- Define a factored probability distribution

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$$

Partition function $= \sum_{\mathbf{Y} \in \mathcal{Y}} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$ **Exponential number of configurations !**

MRFs / CRFs

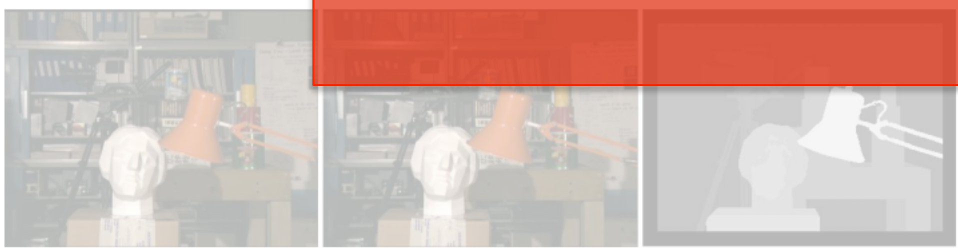
- Several applications, e.g., computer vision

Interactive figure-ground segmentation [Boykov and Jolly, 2001; Boykov and Funka-Lea, 2006]

Surface context [Hoiem et al., 2005]

Semantic labeling [He et al., 2004; Shotton et al., 2006; Gould et al., 2009]

Low-level vision problems



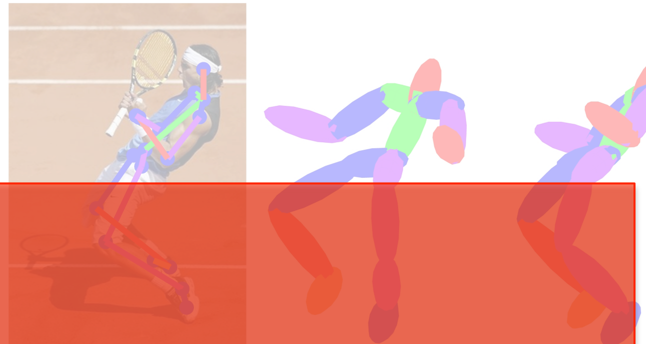
Stereo matching [Kolmogorov and Zabih, 2001; Scharstein and Szeliski, 2002]

Image denoising [Felzenszwalb and Huttenlocher 2004]

MRFs / CRFs

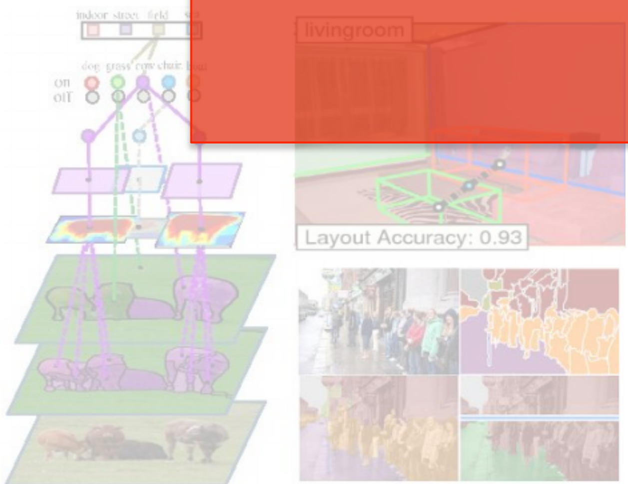
- Several applications, e.g., computer vision

Object detection [Felzenszwalb et al., 2008]



Pose estimation [Akhter and Black, 2015; Ramakrishna et al., 2012]

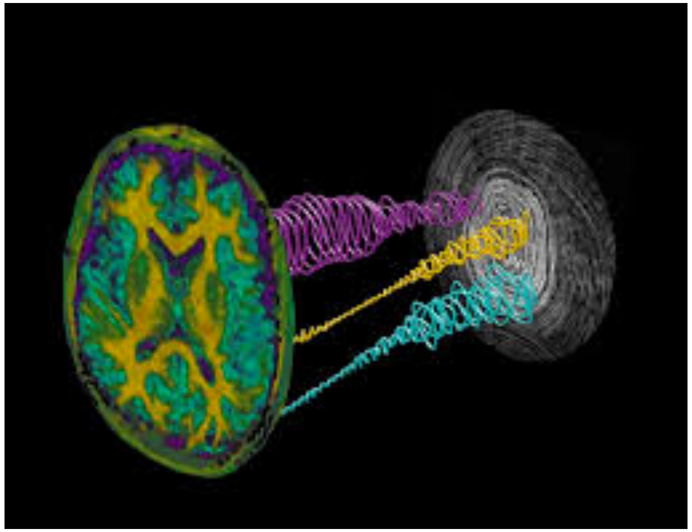
High-level vision problems

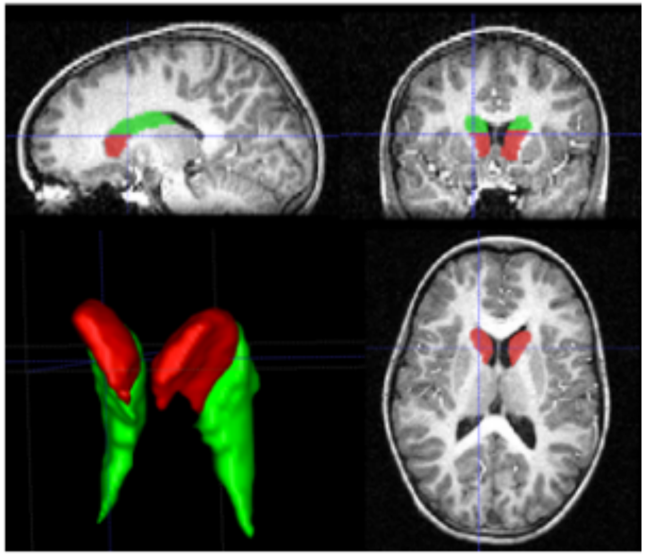


Scene understanding
[Fouhey et al., 2014; Ladicky et al., 2010; Xiao et al., 2013; Yao et al., 2012]

MRFs / CRFs

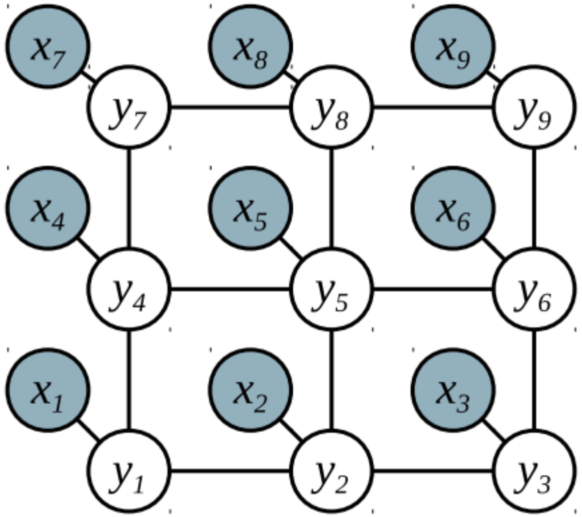
- Several applications, e.g., medical imaging



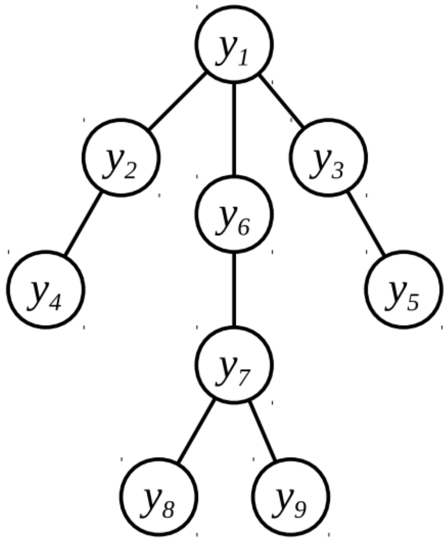


MRFs / CRFs

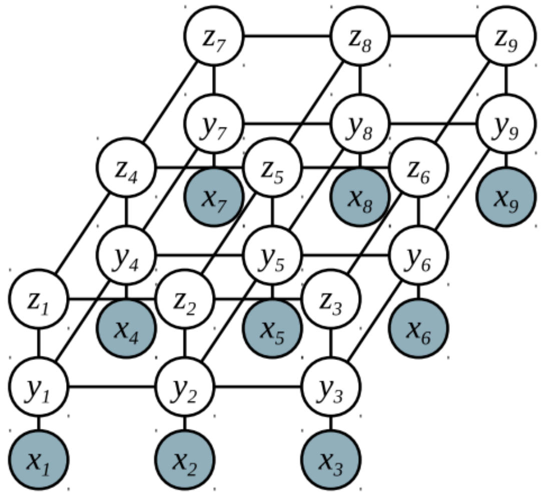
- Inherent in all these problems are graphical models



Pixel labeling



Object detection
Pose estimation



Scene understanding

Maximum a posteriori (MAP) inference

$$
\begin{aligned}
\mathbf{y}^{\star} &= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \frac{1}{Z(\mathbf{X})} \prod_{c} \Psi_c(\mathbf{Y}_c; \mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \log \left(\frac{1}{Z(\mathbf{X})} \prod_{c} \Psi_c(\mathbf{Y}_c; \mathbf{X}) \right) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_{c} \log \Psi_c(\mathbf{Y}_c; \mathbf{X}) - \log Z(\mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_{c} \log \Psi_c(\mathbf{Y}_c; \mathbf{X})
\end{aligned}
$$

$-E(\mathbf{Y}; \mathbf{X})$

Maximum a posteriori (MAP) inference

$$\mathbf{y}^\star = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x}) = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_c \log \Psi_c(\mathbf{Y}_c; \mathbf{X})$$

$$= \operatorname*{argmin}_{\mathbf{y}\in\mathcal{Y}} E(\mathbf{y}; \mathbf{x})$$

MAP inference ⇔ Energy minimization

The energy function is $E(\mathbf{Y}; \mathbf{X}) = \sum_c \psi_c(\mathbf{Y}_c; \mathbf{X})$

Clique potential

where $\psi_c(\cdot) = -\log \Psi_c(\cdot)$

Clique potentials

- Defines a mapping from an assignment of random variables to a real number

$$\psi_c : \mathcal{Y}_c \times \mathcal{X} \rightarrow \mathbb{R}$$

- Encodes a preference for assignments to the random variables (lower is better)

- Parameterized as $\psi_c(\mathbf{y}_c; \mathbf{x}) = \mathbf{w}_c^T \phi_c(\mathbf{y}_c; \mathbf{x})$

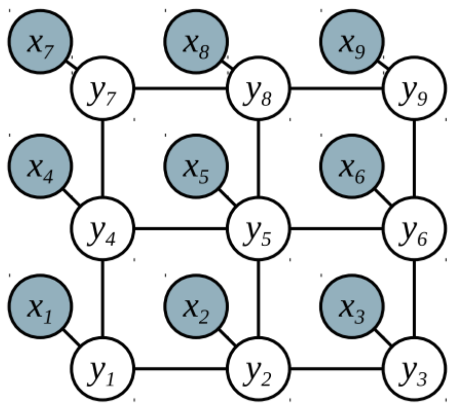
Parameters

Clique potentials

- Arity

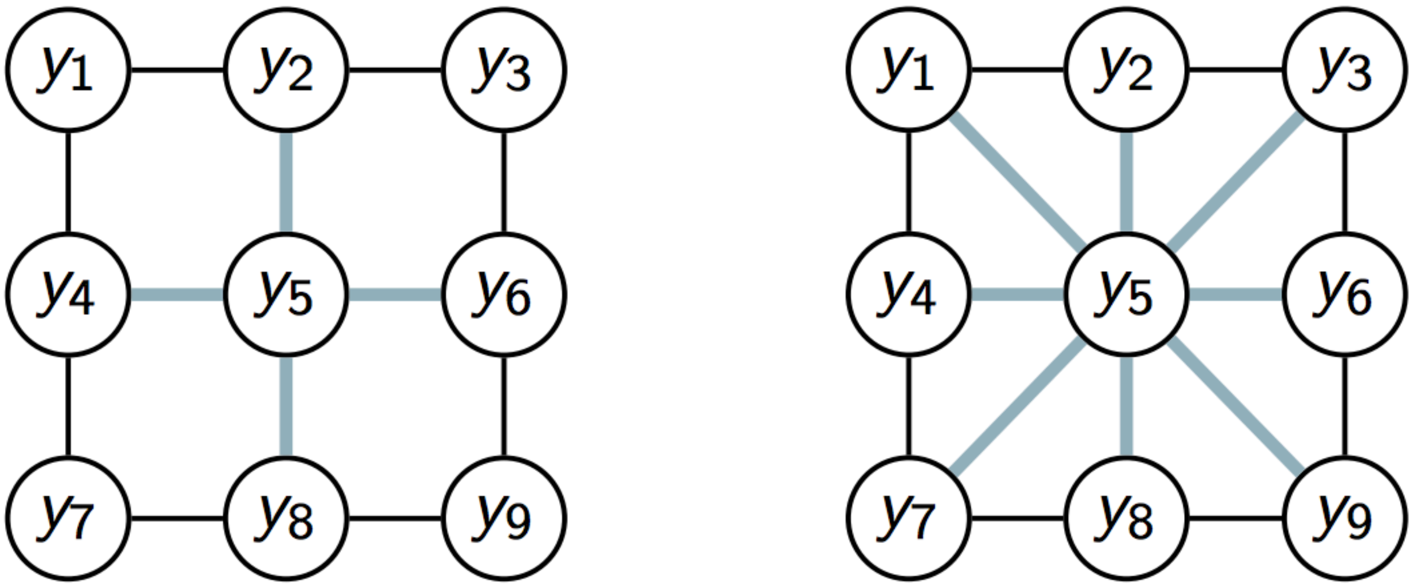
$$E(\mathbf{y};\mathbf{x}) = \sum_{c} \psi_c(\mathbf{y}_c;\mathbf{x})$$

$$= \underbrace{\sum_{i\in\mathcal{V}} \psi_i^U(y_i;\mathbf{x})}_{\text{unary}} + \underbrace{\sum_{ij\in\mathcal{E}} \psi_{ij}^P(y_i,y_j;\mathbf{x})}_{\text{pairwise}} + \underbrace{\sum_{c\in\mathcal{C}} \psi_c^H(\mathbf{y}_c;\mathbf{x})}_{\text{higher-order}}.$$



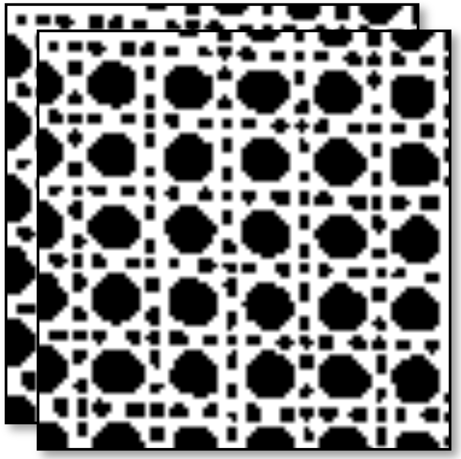
Clique potentials

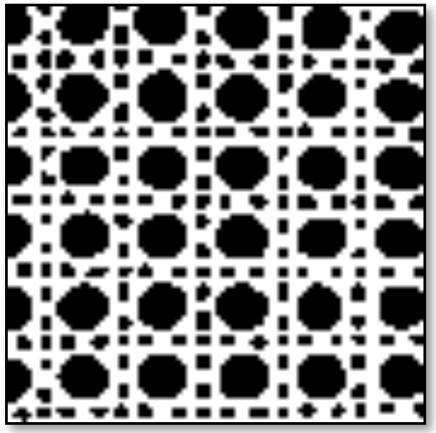
- Arity

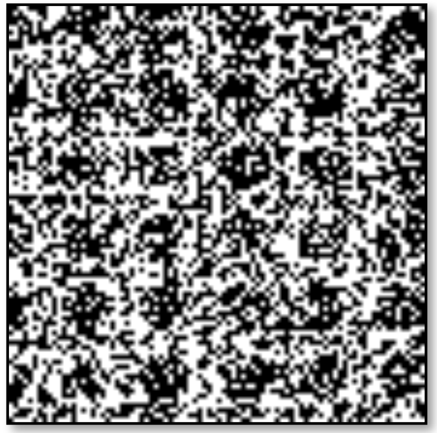


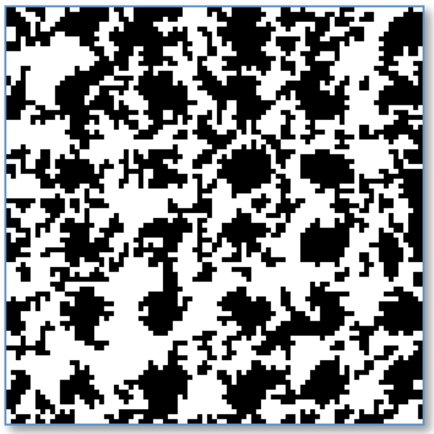
4-connected, $\mathcal{N}_4$ 8-connected, $\mathcal{N}_8$

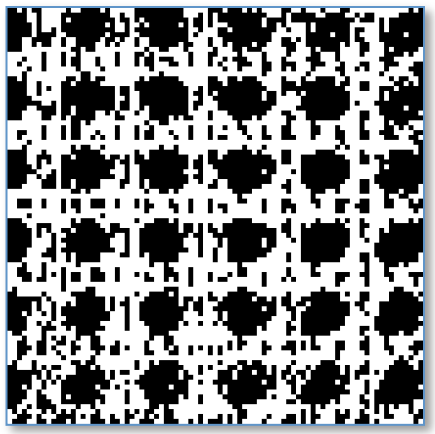
Reason 1: Texture modelling

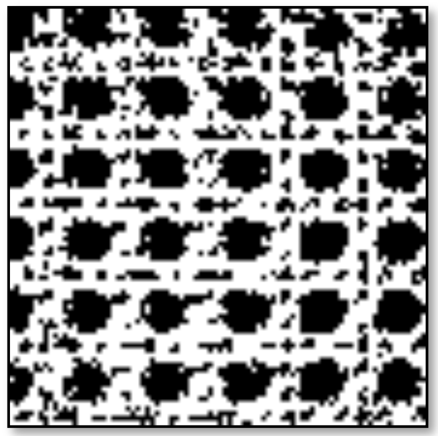

Training images


Test image

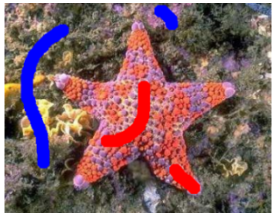

Test image (60% Noise)


Result MRF
4-connected
(neighbours)

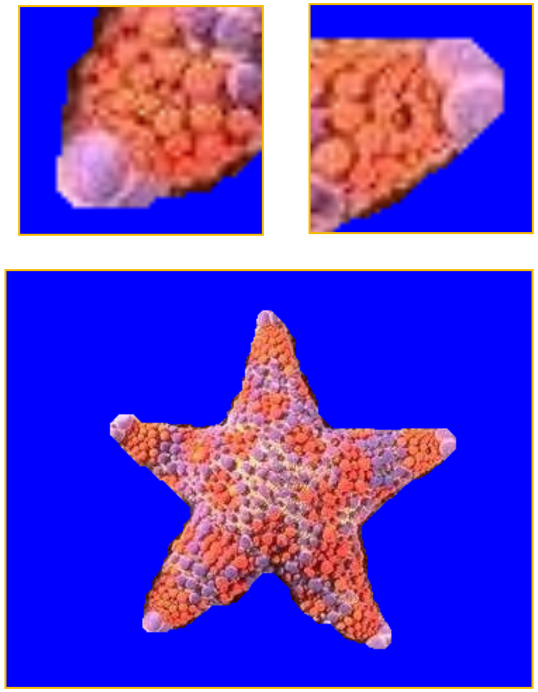

Result MRF
4-connected


Result MRF
9-connected
(7 attractive; 2 repulsive)

Reason2: Discretization artefacts



4-connected
Euclidean



8-connected
Euclidean

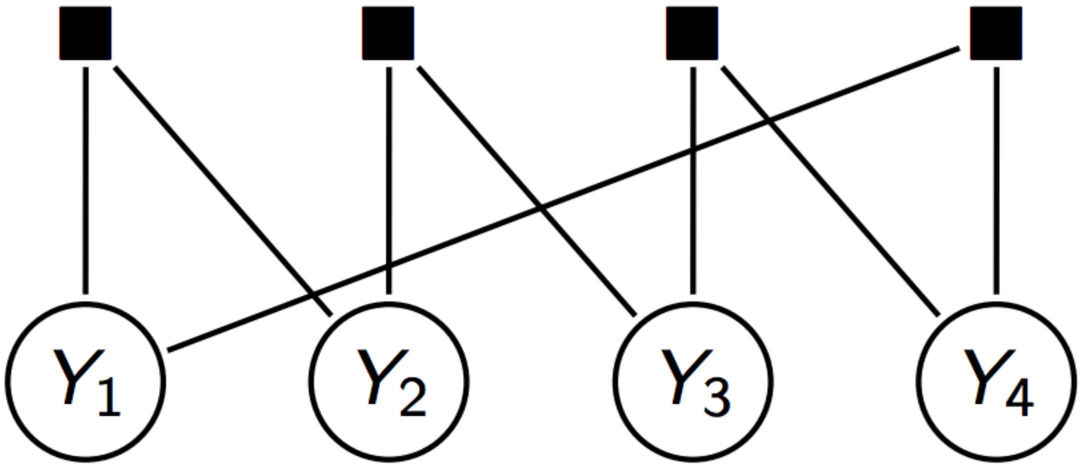
higher-connectivity can model
true Euclidean length

[Boykov et al. ’03; ’05]

Graphical representation

- Example

$$E(\mathbf{y}) = \psi(y_1, y_2) + \psi(y_2, y_3) + \psi(y_3, y_4) + \psi(y_4, y_1)$$

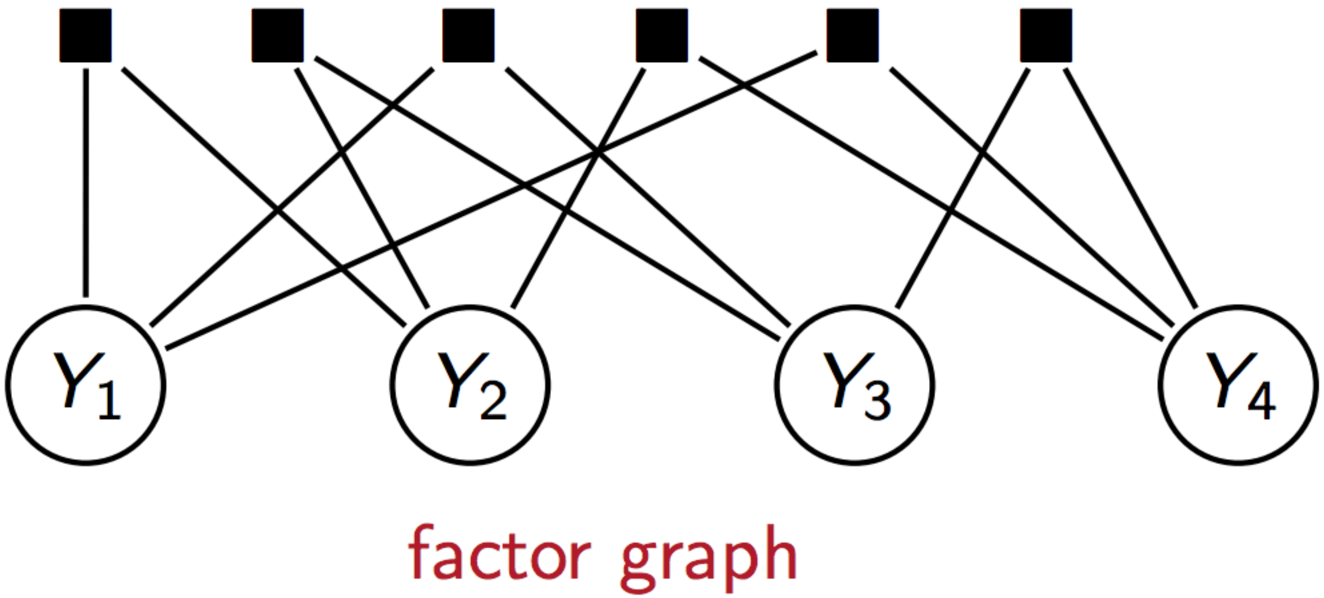


factor graph

Graphical representation

- Example

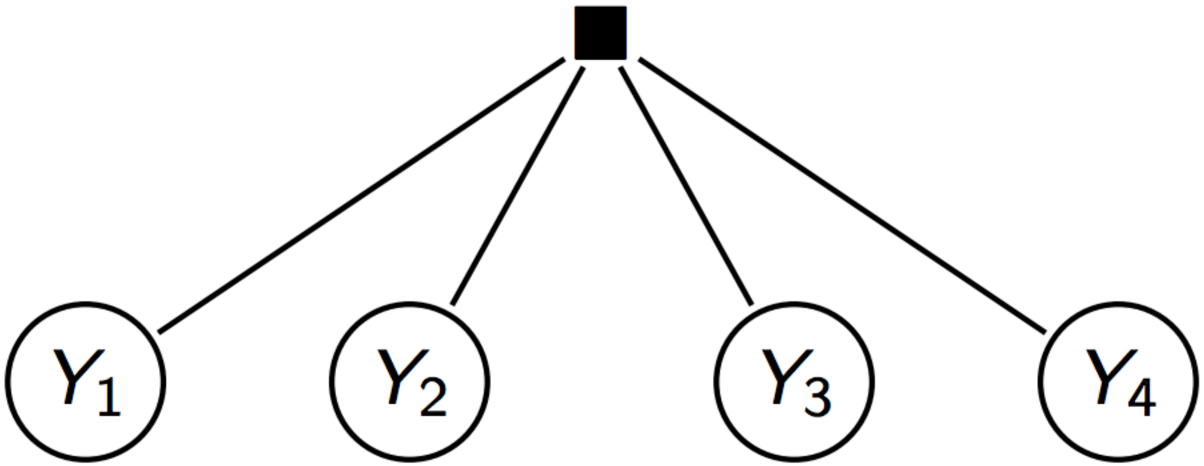
$$E(\mathbf{y}) = \sum_{i,j} \psi(y_i, y_j)$$



Graphical representation

- Example

$$E(\mathbf{y}) = \psi(y_1, y_2, y_3, y_4)$$



factor graph