Graphical Models
Discrete Inference and Learning
Lecture 1

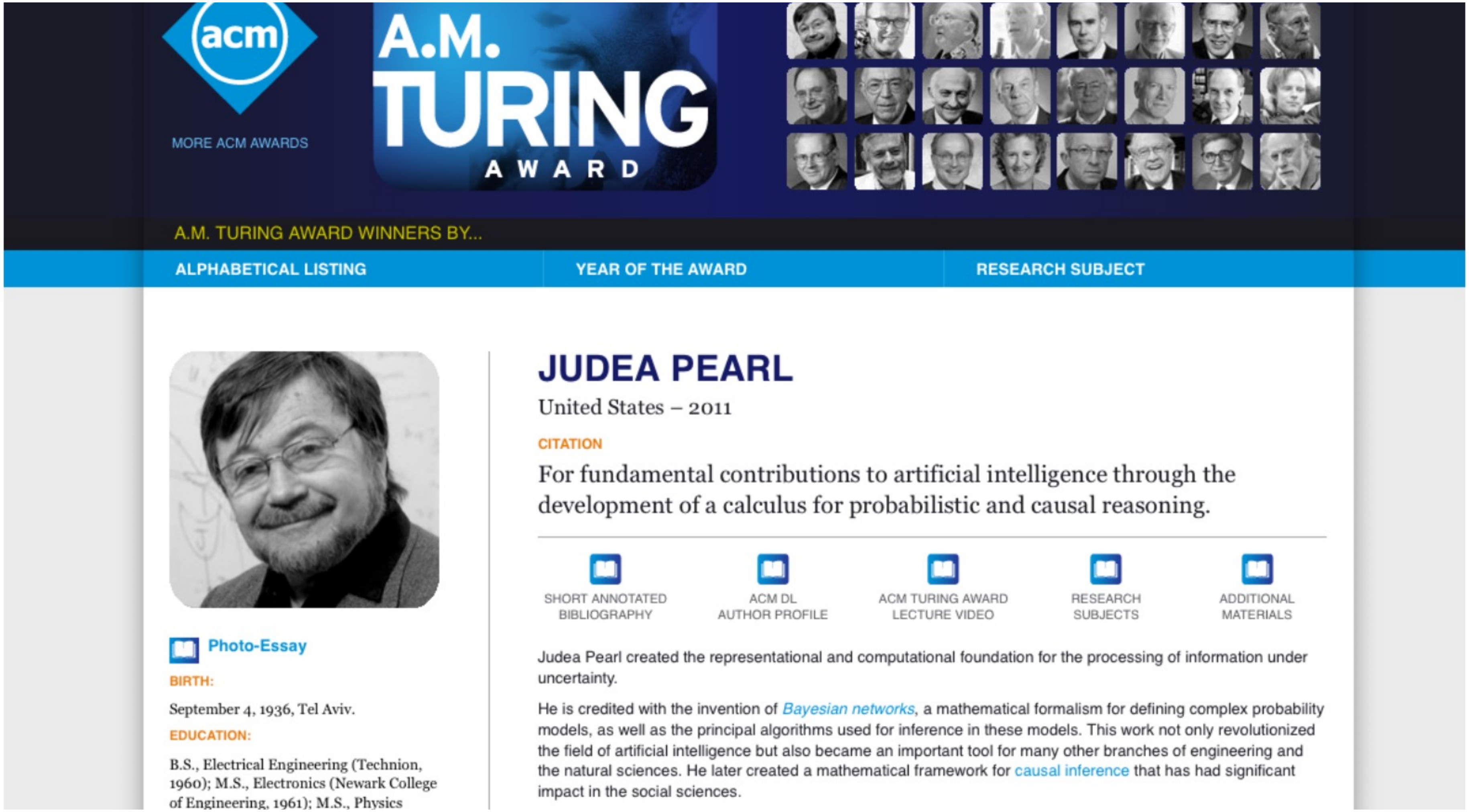
MVA

2024 – 2025

http://thoth.inrialpes.fr/~alahari/disinflearn

Slides based on material from Stephen Gould, Pushmeet Kohli, Nikos Komodakis, M. Pawan Kumar, Carsten Rother, Daphne Koller, Dhruv Batra

Graphical Models ?



Slide courtesy: Dhruv Batra

What this class is about?

- Making **global** predictions from **local** observations

Inference

- Learning such models from large quantities of data

Learning

Motivation

- Consider the example of medical diagnosis

Predisposing factors
Symptoms
Test results

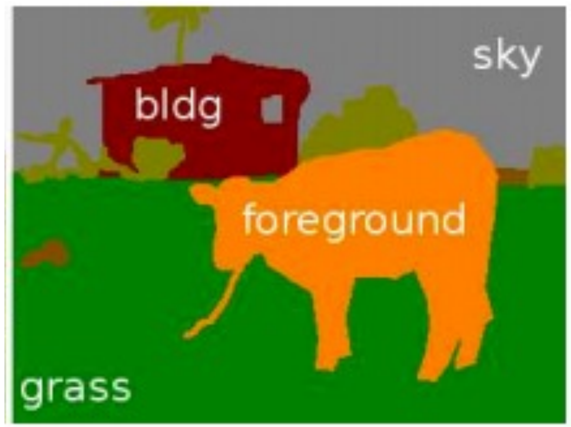
↓

Diseases
Treatment outcomes

Slide inspired by PGM course, Daphne Koller

Motivation

- A very different example: image segmentation



e.g., [He et al., 2004; Shotton et al., 2006; Gould et al., 2009]

Millions of pixels
Colours / features

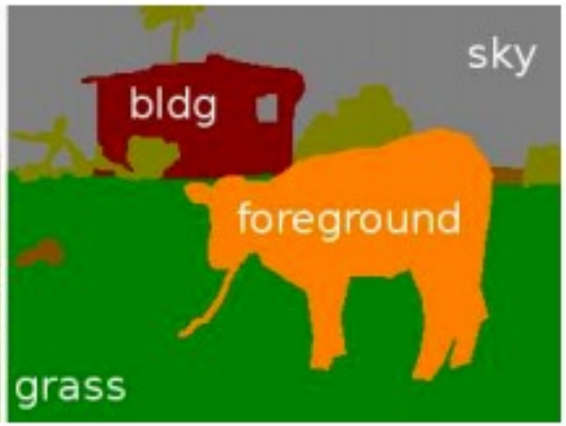
↓

Pixel labels
{building, grass, cow, sky}

Slide inspired by PGM course, Daphne Koller

Motivation

- What do these two problems have in common?



Slide inspired by PGM course, Daphne Koller

Motivation

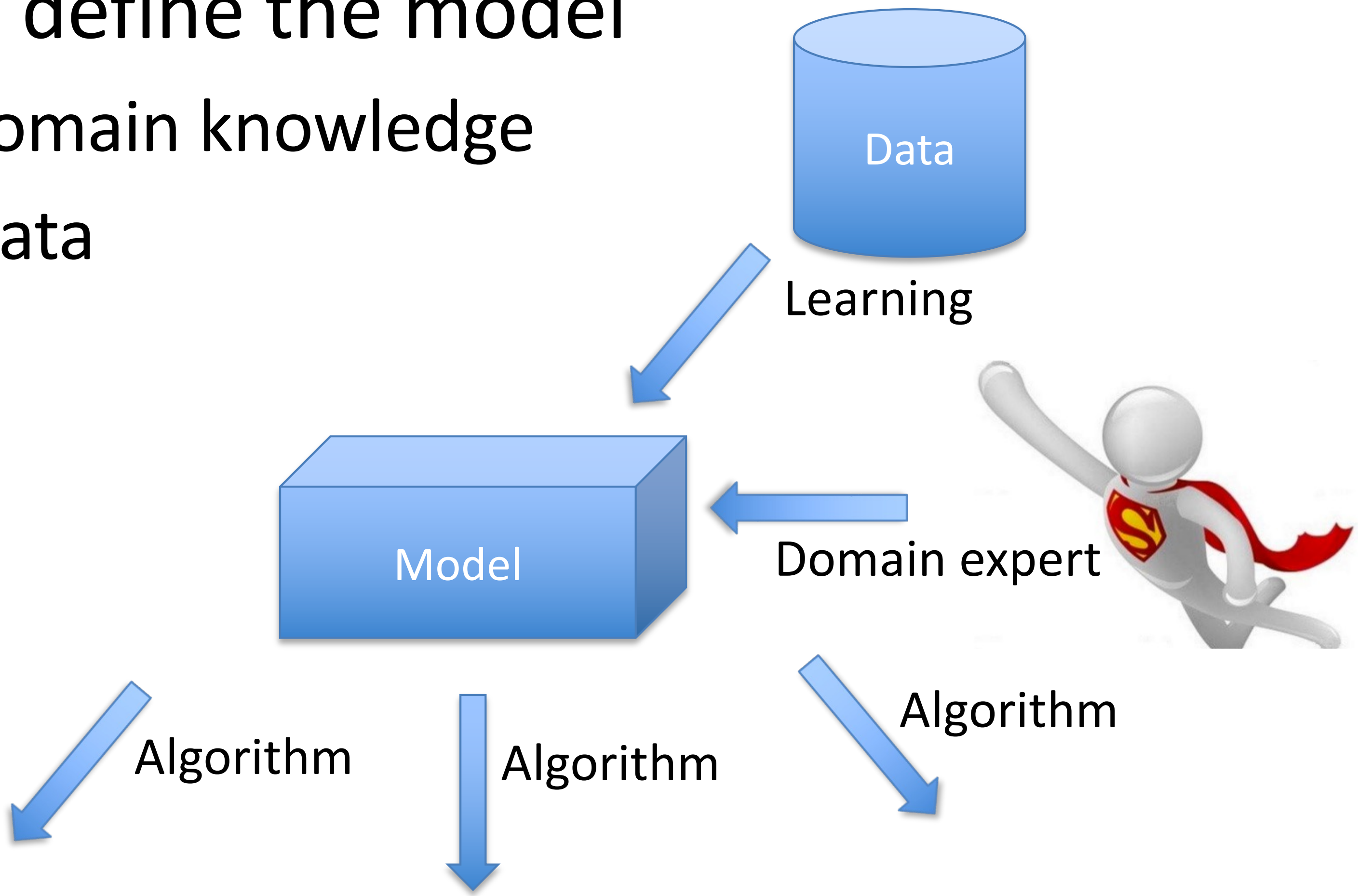
- What do these two problems have in common?
  - Many variables
  - Uncertainty about the correct answer

Graphical Models (or Probabilistic Graphical Models) provide a framework to address these problems

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- First, it is a model: a declarative representation
- Can also define the model
  - with domain knowledge
  - from data



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(Probabilistic) Graphical Models

- Why probabilistic ?
- To model uncertainty
- Uncertainty due to:
  - Partial knowledge of state of the world
  - Noisy observations
  - Phenomena not observed by the model
  - Inherent stochasticity

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- Probability theory provides
  - Standalone representation with clear semantics
  - Reasoning patterns (conditioning, decision making)
  - Learning methods

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

- Why graphical ?
- Intersection of ideas from probability theory and computer science
  - To represent large number of variables

Predisposing factors
Symptoms
Test results

Millions of pixels
Colours / features

Random variables $Y_1, Y_2, \ldots, Y_n$

Goal: capture uncertainty through joint distribution $P(Y_1,\ldots,Y_n)$

Slide inspired by PGM course, Daphne Koller

(Probabilistic) Graphical Models

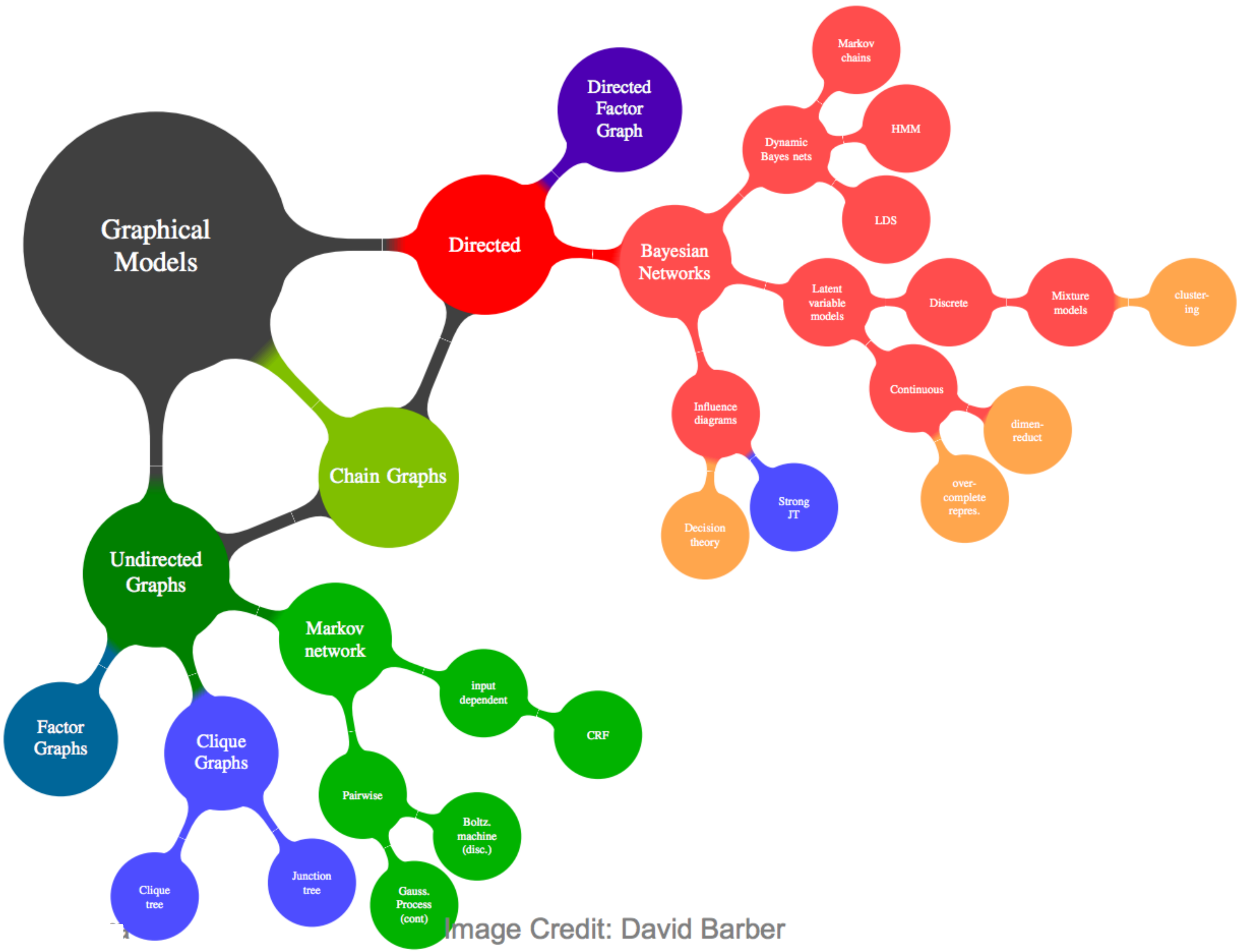
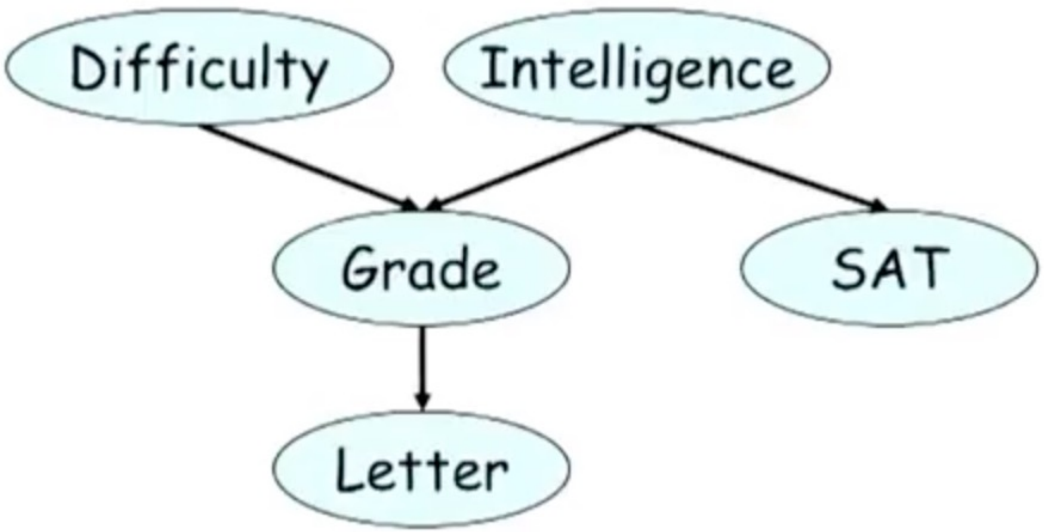


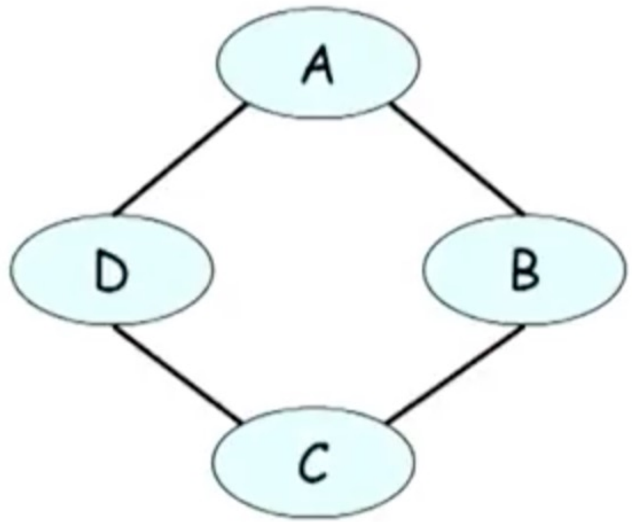
Image Credit: David Barber

(Probabilistic) Graphical Model

- Examples



Bayesian network
(directed graph)

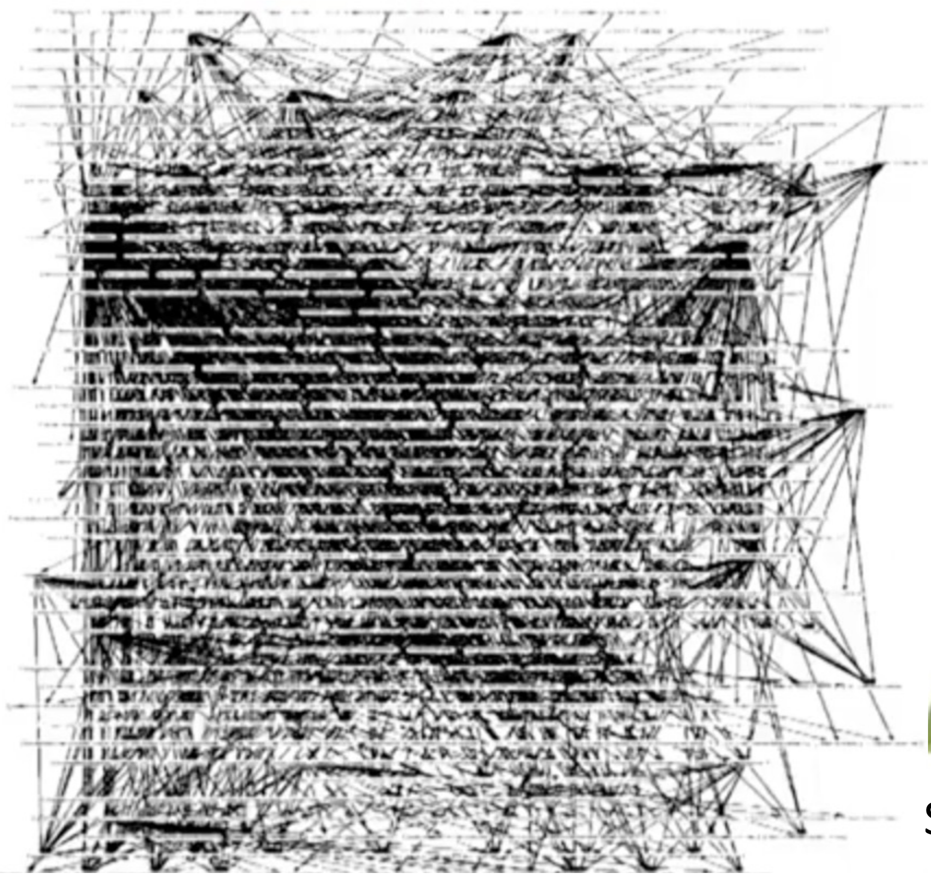


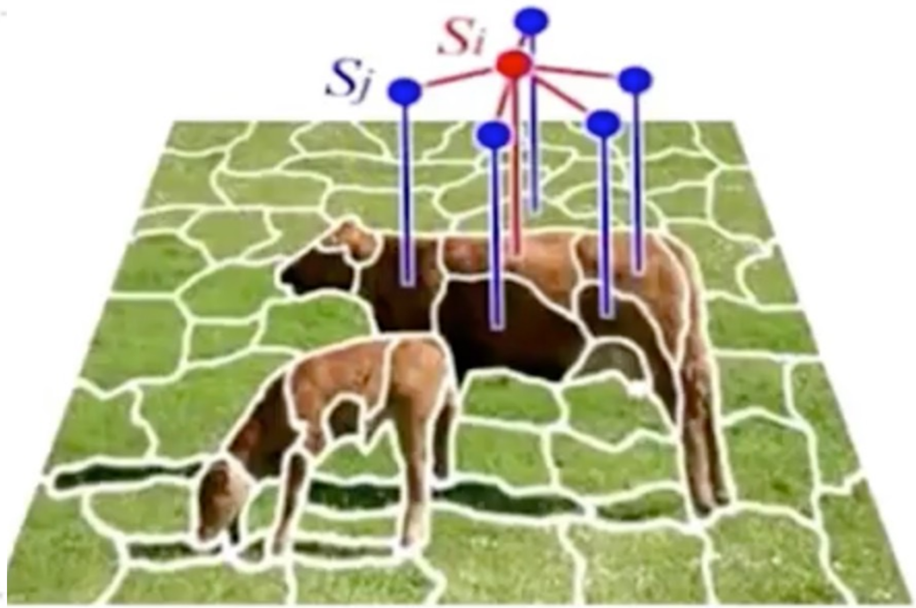
Markov network
(undirected graph)

Figure courtesy: D. Koller

(Probabilistic) Graphical Model

- Examples





Segmentation network (Courtesy D. Koller)

Diagnosis network: Pradhan et al., UAI'94

(Probabilistic) Graphical Model

- Intuitive & compact data structure
- Efficient reasoning through general-purpose algorithms
- Sparse parameterization
  - Through expert knowledge, or
  - Learning from data

Slide inspired by PGM course, Daphne Koller

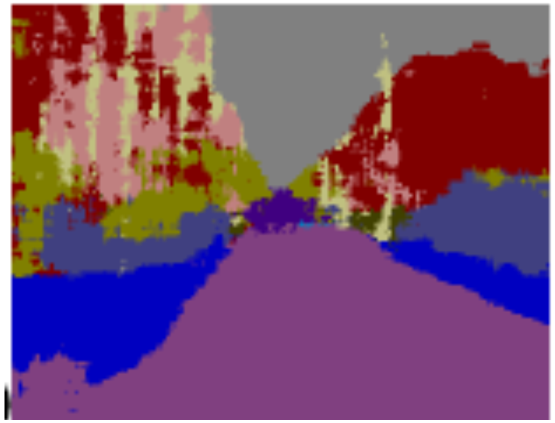
(Probabilistic) Graphical Model

- Many many applications
  - Medical diagnosis
  - Fault diagnosis
  - Natural language processing
  - Traffic analysis
  - Social network models
  - Message decoding
  - Computer vision: segmentation, 3D, pose estimation
  - Speech recognition
  - Robot localization & mapping

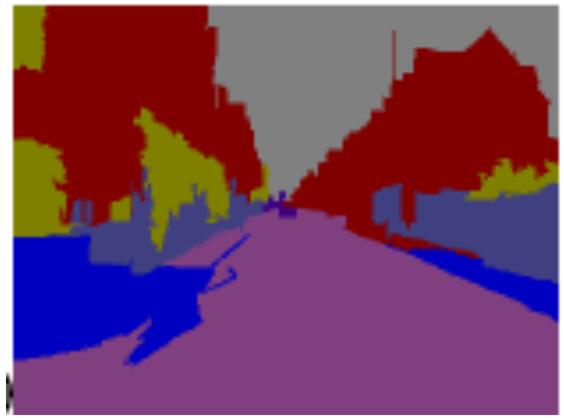
Slide courtesy: PGM course, Daphne Koller

Image segmentation

Image



No graphical model

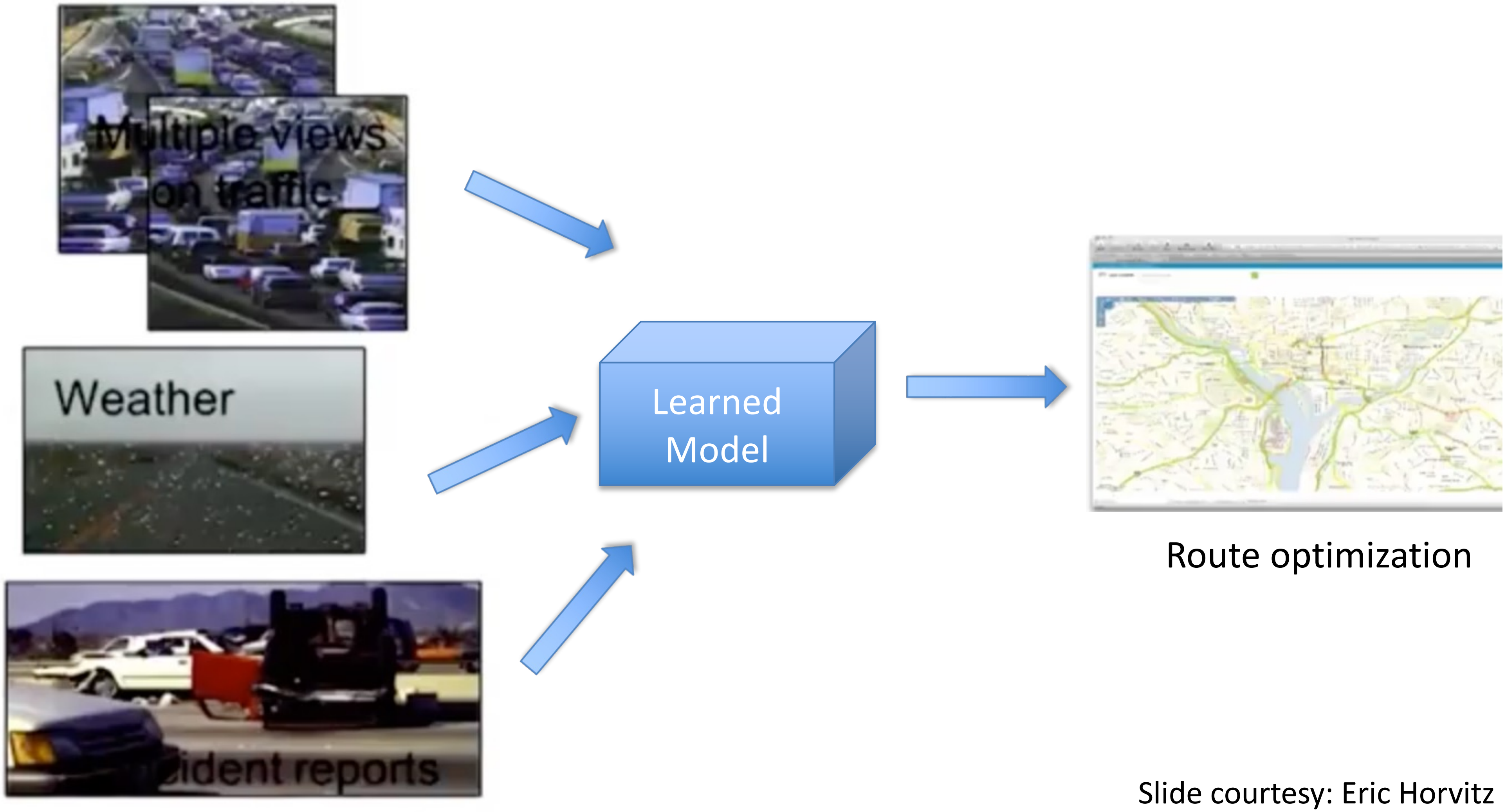


With graphical model

Sturgess et al., 2009

Multi-sensor integration: Traffic

- Learn from historical data to make predictions



Slide courtesy: Eric Horvitz

Going global: Local ambiguity

- Text recognition

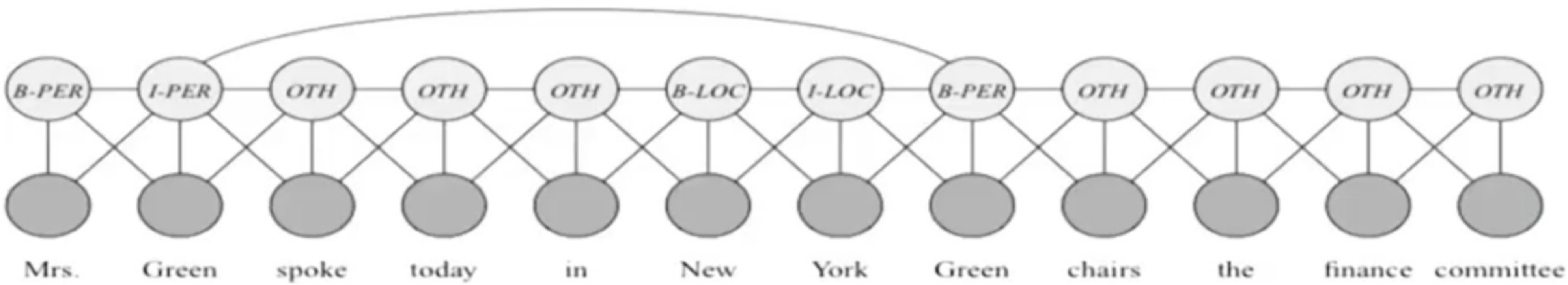
Smyth et al., 1994

Slide courtesy: Dhruv Batra

Going global: Local ambiguity

- Textual information extraction

e.g., Mrs. Green spoke today in New York. Green chairs the financial committee.



Slide courtesy: PGM course, Daphne Koller

Overview

- Representation
  - How do we store $P(Y_1,\ldots Y_n)$
  - Directed and undirected (model implications/assumptions)
- Inference
  - Answer questions with the model
  - Exact and approximate (marginal/most probable estimate)
- Learning
  - What model is right for data
  - Parameters and structure

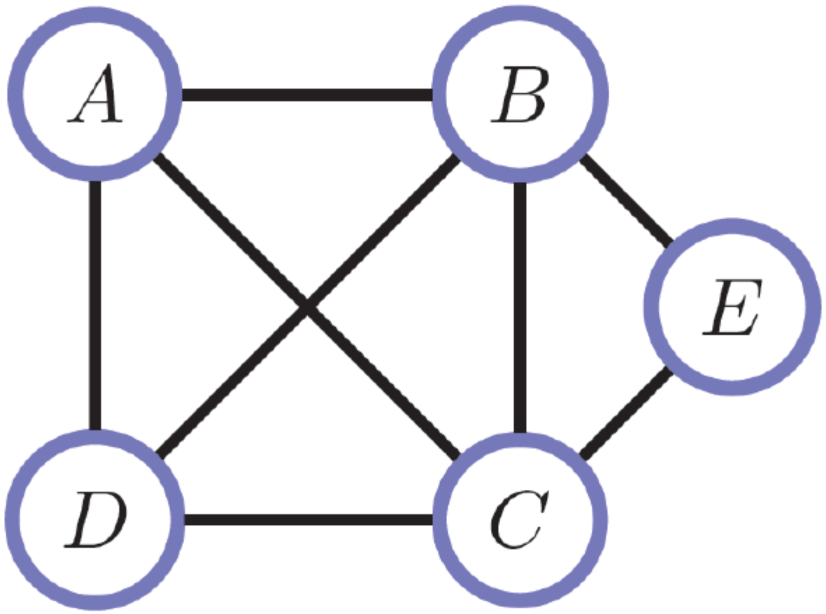
Slide inspired by D. Batra, D. Koller 's courses

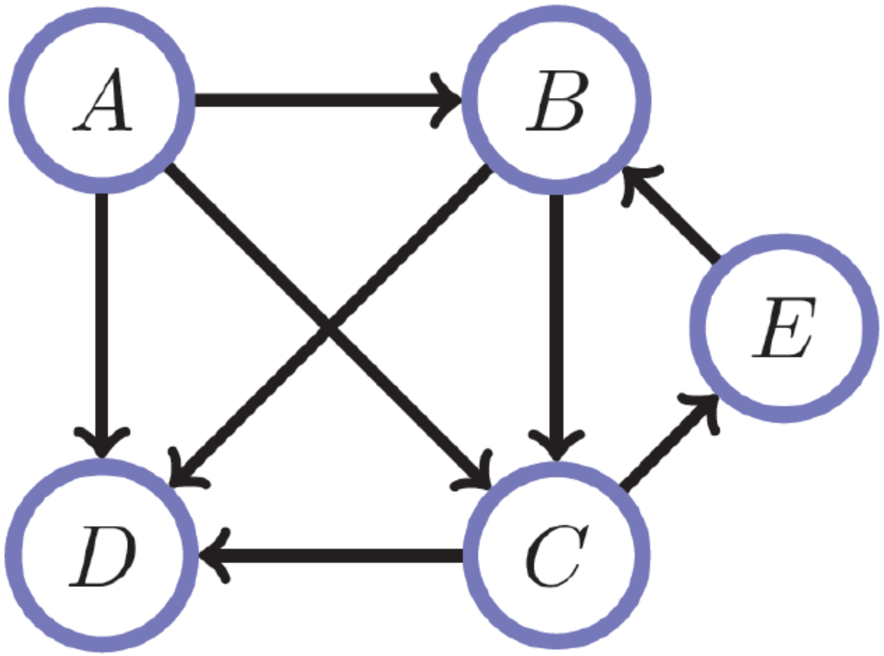
First, a recap of basics

Graphs

- Concepts
  - Definition of G
  - Vertices/Nodes
  - Edges
  - Directed vs Undirected
  - Neighbours vs Parent/Child
  - Degree vs In/Out degree
  - Walk vs Path vs Cycle

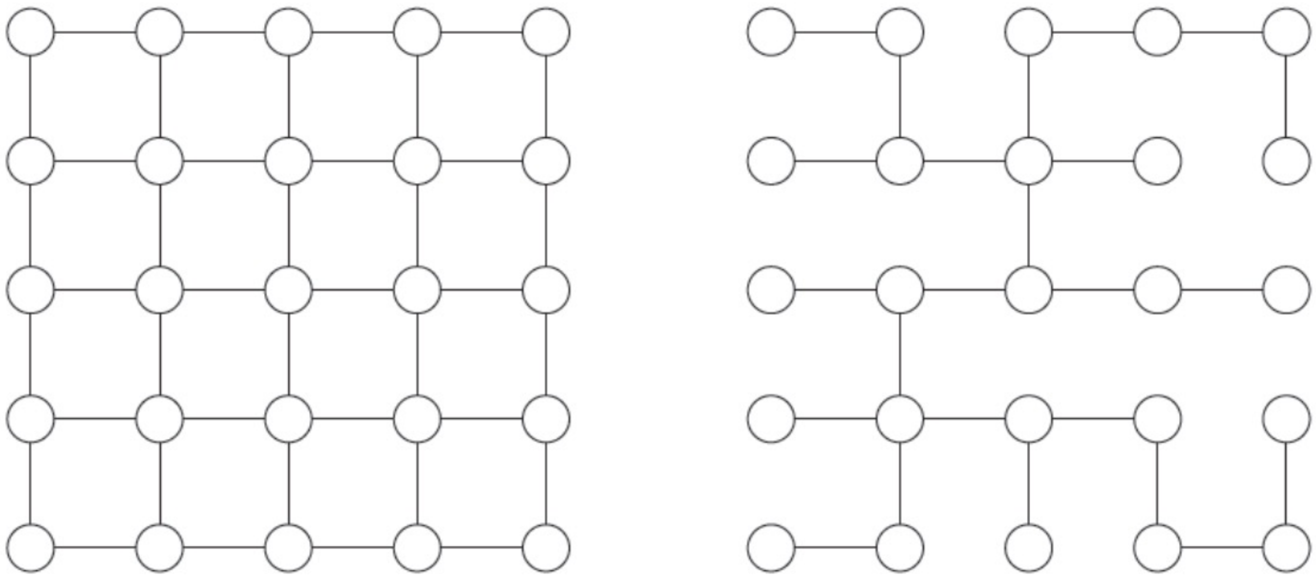
Graphs





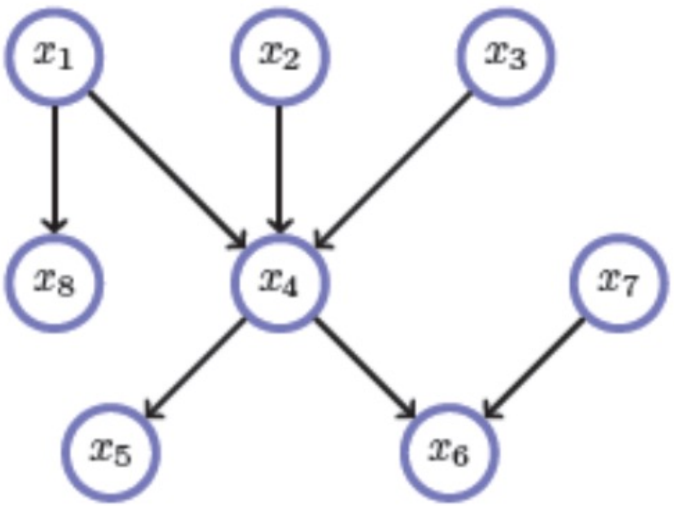
Special graphs

- Trees: undirected graph, no cycles
- Spanning tree: Same set of vertices, but subset of edges, connected and no cycles



Slide courtesy: D. Batra

Directed acyclic graphs (DAGs)



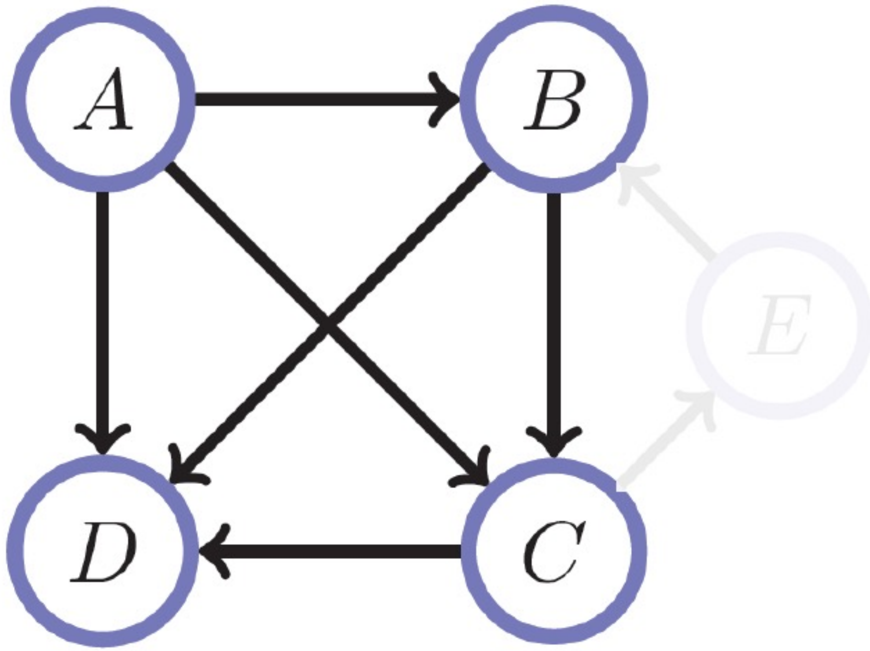


Figure courtesy: D. Batra

Joint distribution

- 3 variables
  - Intelligence (I)
  - Difficulty (D)
  - Grade (G)

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^0	g^2	0.168
i^0	d^0	g^3	0.126
i^0	d^1	g^1	0.009
i^0	d^1	g^2	0.045
i^0	d^1	g^3	0.126
i^1	d^0	g^1	0.252
i^1	d^0	g^2	0.0224
i^1	d^0	g^3	0.0056
i^1	d^1	g^1	0.06
i^1	d^1	g^2	0.036
i^1	d^1	g^3	0.024

Example courtesy: PGM course, Daphne Koller

Conditioning

- Condition on g^1

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^0	g^2	0.168
i^0	d^0	g^3	0.126
i^0	d^1	g^1	0.009
i^0	d^1	g^2	0.045
i^0	d^1	g^3	0.126
i^1	d^0	g^1	0.252
i^1	d^0	g^2	0.0224
i^1	d^0	g^3	0.0056
i^1	d^1	g^1	0.06
i^1	d^1	g^2	0.036
i^1	d^1	g^3	0.024

Example courtesy: PGM course, Daphne Koller

Conditioning

- $P(Y = y \mid X = x)$
- Informally,
  - What do you believe about Y=y when I tell you X=x ?

- P(France wins Euro 2024) ?
- What if I tell you:
  - France almost won the world cup 2022
  - Hasn't had catastrophic results since ☺

Slide based on D. Batra's course

Conditioning: Reduction

- Condition on g^1

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.06

Example courtesy: PGM course, Daphne Koller

Conditioning: Renormalization

I	D	G	Prob.
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.06

$P(I, D, g^1)$

Unnormalized measure

→

I	D	Prob.
i^0	d^0	0.282
i^0	d^1	0.02
i^1	d^0	0.564
i^1	d^1	0.134

$P(I, D \mid g^1)$

Example courtesy: PGM course, Daphne Koller

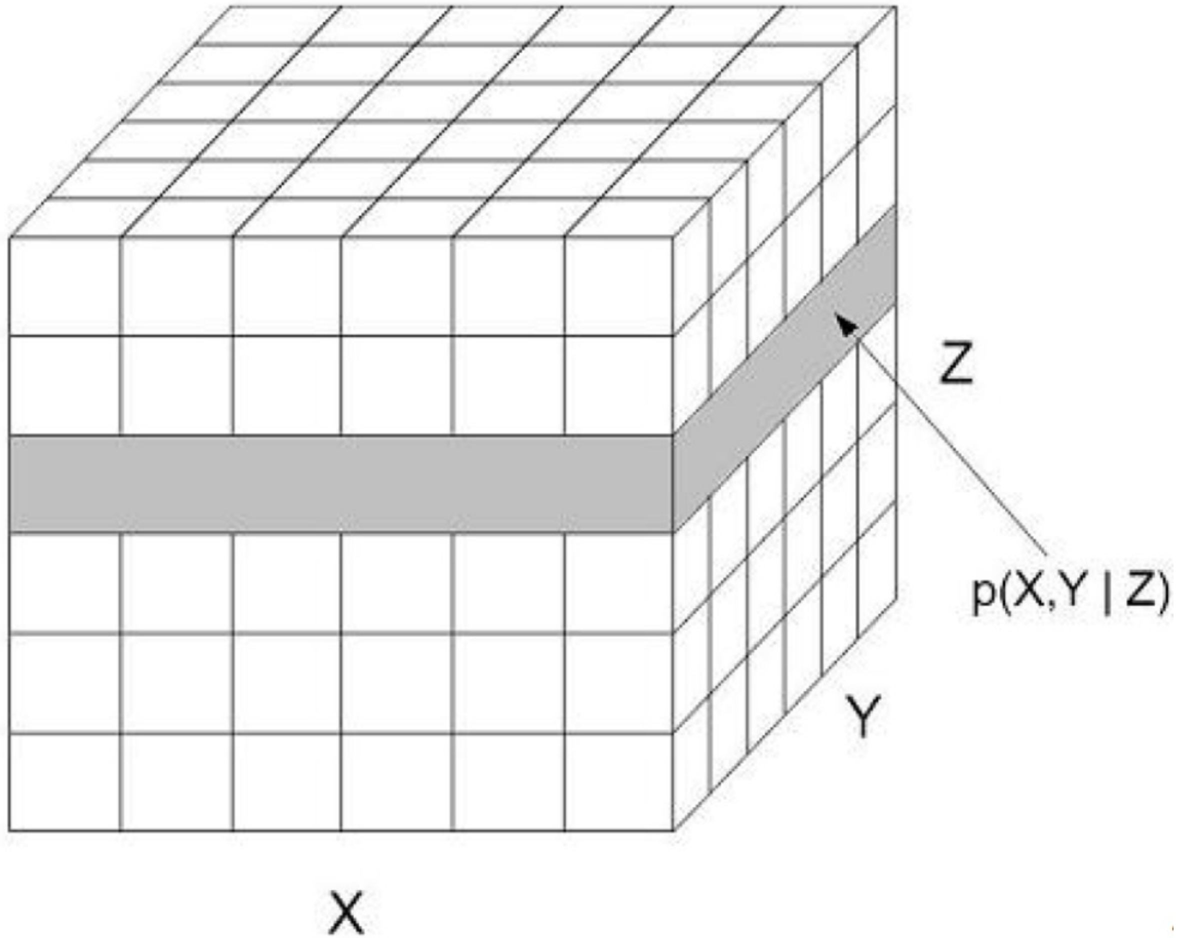
Conditional probability distribution

- Example $P(G \mid I, D)$

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^1,d^0	0.9	0.08	0.02
i^1,d^1	0.5	0.3	0.2

Example courtesy: PGM course, Daphne Koller

Conditional probability distribution



$$p(x, y \mid Z = z) = \frac{p(x, y, z)}{p(z)}$$

Slide courtesy: Erik Sudderth

Marginalization

P(I,D)

I	D	Prob.
i^0	d^0	0.282
i^0	d^1	0.02
i^1	d^0	0.564
i^1	d^1	0.134

Marginalize I

D	Prob.
d^0	0.846
d^1	0.154

Example courtesy: PGM course, Daphne Koller

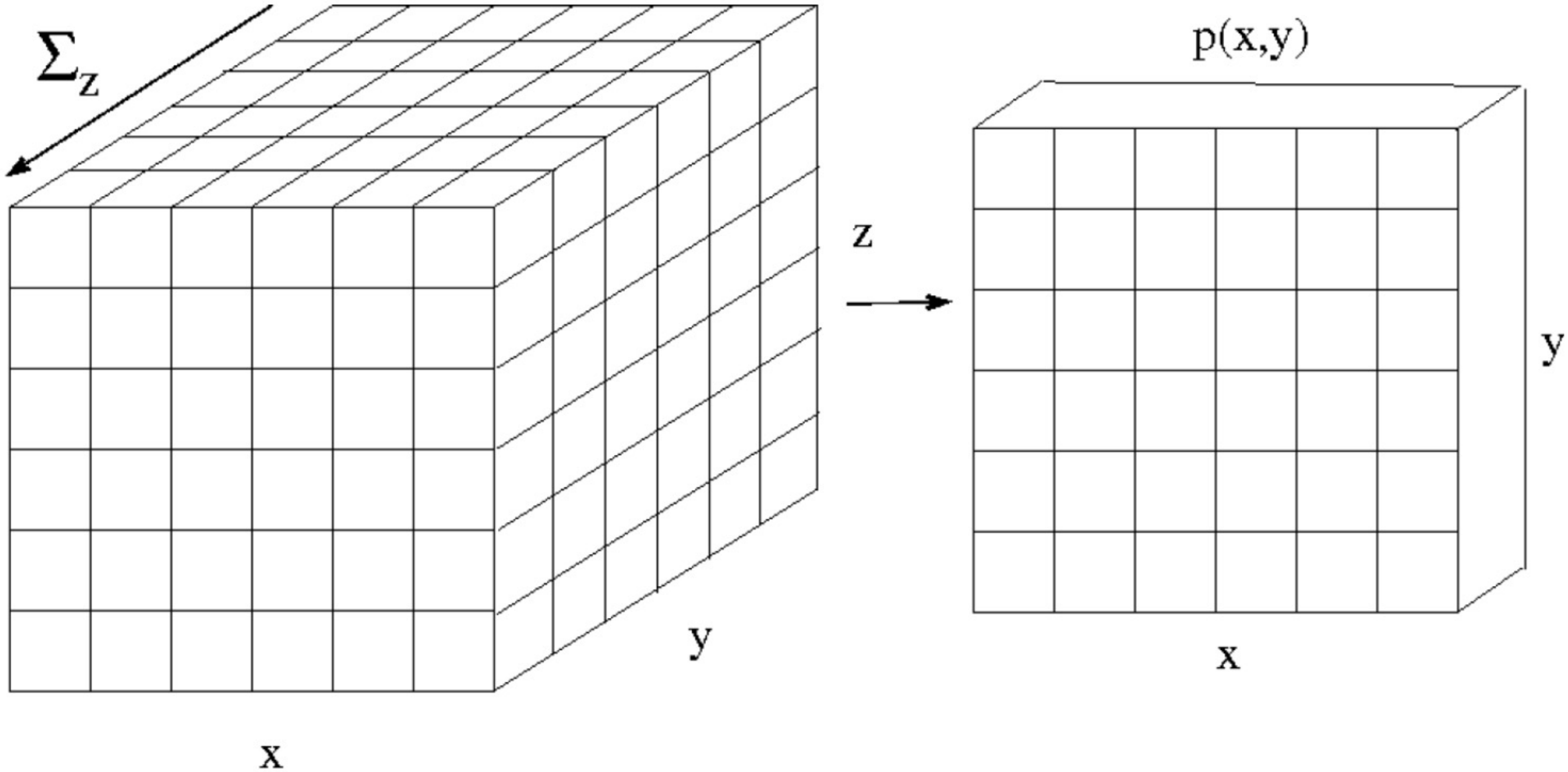
Marginalization

- Events
  - P(A) = P(A and B) + P(A and not B)

- Random variables
  - $P(X = x) = \sum_y P(X = x, Y = y)$

Slide courtesy: Dhruv Batra

Marginalization



$$p(x,y) = \sum_{z \in \mathcal{Z}} p(x,y,z)$$

$$p(x) = \sum_{y \in \mathcal{Y}} p(x,y)$$

Slide courtesy: Erik Sudderth

Factors

- A factor $\Phi(Y_1,\ldots,Y_k)$

  Φ: Val$(Y_1,\ldots,Y_k) \rightarrow \mathbb{R}$

- Scope = $\{Y_1,\ldots,Y_k\}$

General factors

- Not necessarily for probabilities

A	B	ϕ
a^0	b^0	30
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10

Example courtesy: PGM course, Daphne Koller

Factor product

a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.1
a^2	b^2	0
a^3	b^1	0.3
a^3	b^2	0.9

b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

a^1	b^1	c^1	$0.5 \cdot 0.5 = 0.25$
a^1	b^1	c^2	$0.5 \cdot 0.7 = 0.35$
a^1	b^2	c^1	$0.8 \cdot 0.1 = 0.08$
a^1	b^2	c^2	$0.8 \cdot 0.2 = 0.16$
a^2	b^1	c^1	$0.1 \cdot 0.5 = 0.05$
a^2	b^1	c^2	$0.1 \cdot 0.7 = 0.07$
a^2	b^2	c^1	$0 \cdot 0.1 = 0$
a^2	b^2	c^2	$0 \cdot 0.2 = 0$
a^3	b^1	c^1	$0.3 \cdot 0.5 = 0.15$
a^3	b^1	c^2	$0.3 \cdot 0.7 = 0.21$
a^3	b^2	c^1	$0.9 \cdot 0.1 = 0.09$
a^3	b^2	c^2	$0.9 \cdot 0.2 = 0.18$

Example courtesy: PGM course, Daphne Koller

Factor marginalization

a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.05
a^2	b^1	c^2	0.07
a^2	b^2	c^1	0
a^2	b^2	c^2	0
a^3	b^1	c^1	0.15
a^3	b^1	c^2	0.21
a^3	b^2	c^1	0.09
a^3	b^2	c^2	0.18

a^1	c^1	0.33
a^1	c^2	0.51
a^2	c^1	0.05
a^2	c^2	0.07
a^3	c^1	0.24
a^3	c^2	0.39

Example courtesy: PGM course, Daphne Koller

Factor reduction

a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.05
a^2	b^1	c^2	0.07
a^2	b^2	c^1	0
a^2	b^2	c^2	0
a^3	b^1	c^1	0.15
a^3	b^1	c^2	0.21
a^3	b^2	c^1	0.09
a^3	b^2	c^2	0.18

a^1	b^1	c^1	0.25
a^1	b^2	c^1	0.08
a^2	b^1	c^1	0.05
a^2	b^2	c^1	0
a^3	b^1	c^1	0.15
a^3	b^2	c^1	0.09

Why factors ?

- Building blocks for defining distributions in high-dimensional spaces

- Set of basic operations for manipulating these distributions

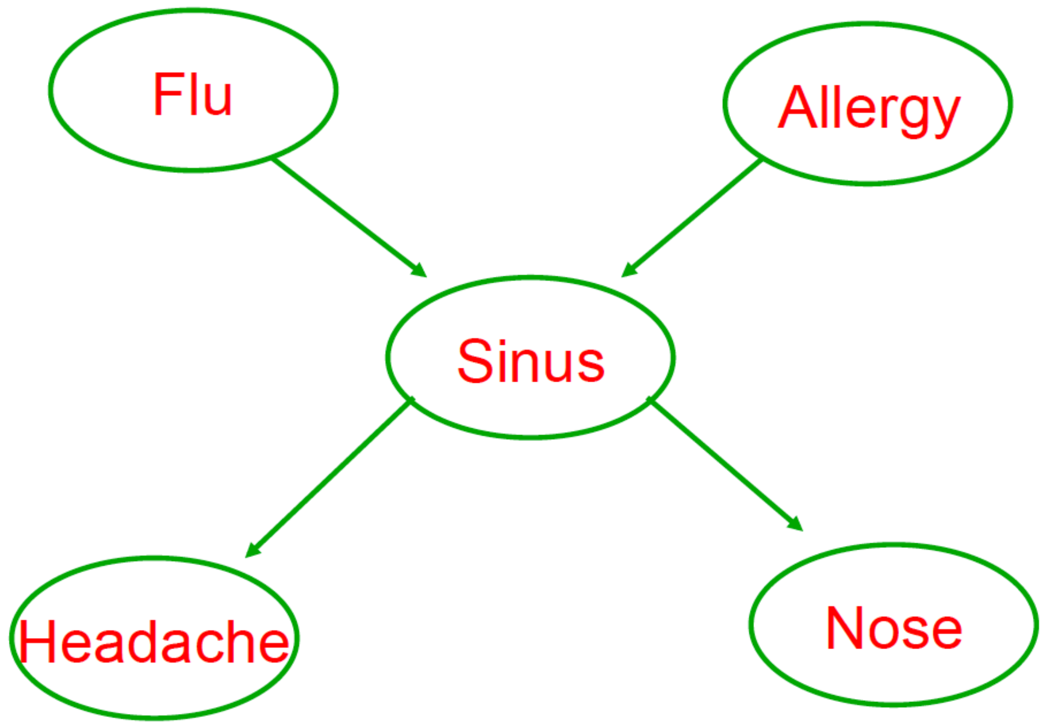
Bayesian Networks

- DAGs
  - nodes represent variables in the Bayesian sense
  - edges represent conditional dependencies
- Example
  - Suppose that we know the following:
    - The flu causes sinus inflammation
    - Allergies cause sinus inflammation
    - Sinus inflammation causes a runny nose
    - Sinus inflammation causes headaches
  - How are these connected ?

Slide courtesy: Dhruv Batra

Bayesian Networks

- Example



Slide courtesy: Dhruv Batra

Bayesian Networks

- A general Bayes net
  - Set of random variables
  - DAG: encodes independence assumptions
  - Conditional probability trees
  - Joint distribution

$$P(Y_1,\ldots,Y_n) = \prod_{i=1}^{n} P(Y_i \mid \mathrm{Pa}_{Y_i})$$

Slide courtesy: Dhruv Batra

Bayesian Networks

- A general Bayes net
  - How many parameters ?
    - Discrete variables $Y_1,\ldots,Y_n$
    - Graph: Defines parents of Y_i, i.e., (Pa_{Y_i})
    - CPTs: $P(Y_i|Pa_{Y_i})$

Slide courtesy: Dhruv Batra

Markov nets

- Set of random variables
- Undirected graph
  - Encodes independence assumptions
- Factors

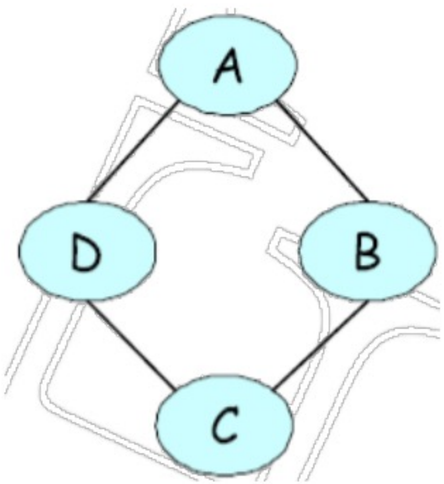
Comparison to Bayesian Nets ?

Slide courtesy: Dhruv Batra

Pairwise MRFs

- Composed of pairwise factors
  - A function of two variables
  - Can also have unary terms

- Example



$\phi_1[A,B]$			$\phi_2[B,C]$			$\phi_3[C,D]$			$\phi_4[D,A]$		
a^0	b^0	30	b^0	c^0	100	c^0	d^0	1	d^0	a^0	100
a^0	b^1	5	b^0	c^1	1	c^0	d^1	100	d^0	a^1	1
a^1	b^0	1	b^1	c^0	1	c^1	d^0	100	d^1	a^0	1
a^1	b^1	10	b^1	c^1	100	c^1	d^1	1	d^1	a^1	100

Slide courtesy: Dhruv Batra

Markov Nets: Computing probabilities

- Can only compute ratio of probabilities directly

$\phi_1[A,B]$			$\phi_2[B,C]$			$\phi_3[C,D]$			$\phi_4[D,A]$		
a^0	b^0	30	b^0	c^0	100	c^0	d^0	1	d^0	a^0	100
a^0	b^1	5	b^0	c^1	1	c^0	d^1	100	d^0	a^1	1
a^1	b^0	1	b^1	c^0	1	c^1	d^0	100	d^1	a^0	1
a^1	b^1	10	b^1	c^1	100	c^1	d^1	1	d^1	a^1	100

- Need to normalize with a **partition function**
  - Hard ! (sum over all possible assignments)
- In Bayesian Nets, can do by multiplying CPTs

Slide courtesy: Dhruv Batra

Markov nets ←→ Factorization

- Given an undirected graph H over variables $Y=\{Y_1,\ldots,Y_n\}$
- A distribution P factorizes over H if there exist
  - Subsets of variables $S^i \subseteq Y$ s.t. S^i are fully-connected in H
  - Non-negative potentials (factors) $\Phi_1(S^1),\ldots, \Phi_m(S^m)$: clique potentials
  - Such that

$$P(Y_1,\ldots,Y_n) = \frac{1}{Z}\prod_{i=1}^{m}\Phi_i(S^i)$$

Slide courtesy: Dhruv Batra

Conditional Markov Random Fields

- Also known as: Markov networks, undirected graphical models, MRFs
- Note: Not making a distinction between CRFs and MRFs
- $\mathbf{X} \in \mathcal{X}$: observed random variables
- $\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathcal{Y}$: output random variables
- $\mathbf{Y}_c$ are subset of variables for clique $c \subseteq \{1, \ldots, n\}$
- Define a factored probability distribution

$$P(\mathbf{Y} \mid \mathbf{X}) = \frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$$

Partition function $= \sum_{\mathbf{Y} \in \mathcal{Y}} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X})$

Exponential number of configurations !

MRFs / CRFs

- Several applications, e.g., computer vision

Interactive figure-ground segmentation [Boykov and Jolly, 2001; Boykov and Funka-Lea, 2006]

Surface context [Hoiem et al., 2005]

Semantic labeling [He et al., 2004; Shotton et al., 2006; Gould et al., 2009]

Low-level vision problems

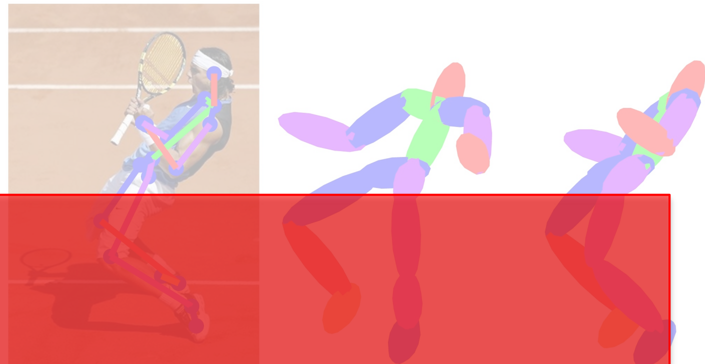
Stereo matching [Kolmogorov and Zabih, 2001; Scharstein and Szeliski, 2002]

Image denoising [Felzenszwalb and Huttenlocher 2004]

MRFs / CRFs

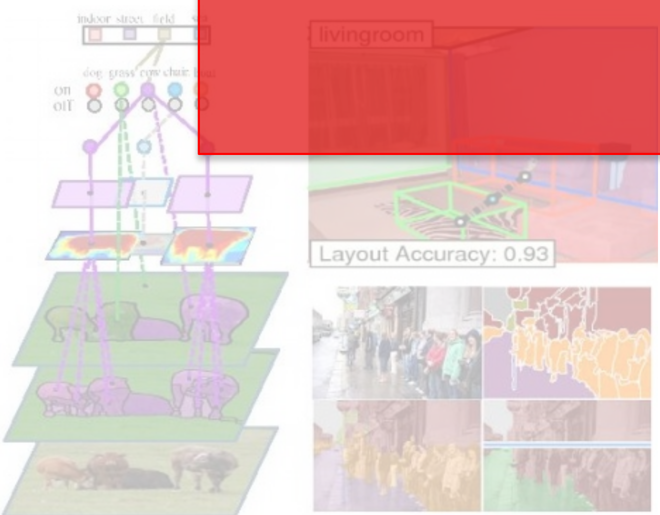
- Several applications, e.g., computer vision

Object detection [Felzenszwalb et al., 2008]



Pose estimation [Zuffi and Black, 2015; Ramakrishna et al., 2012]

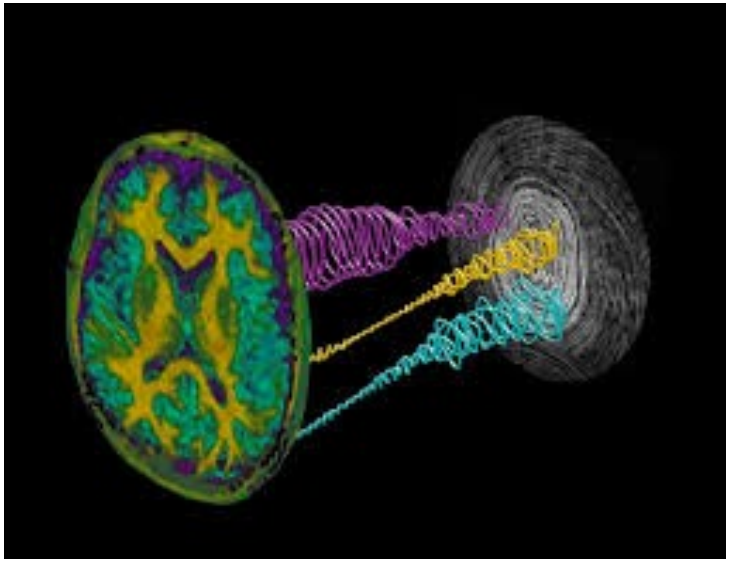
High-level vision problems

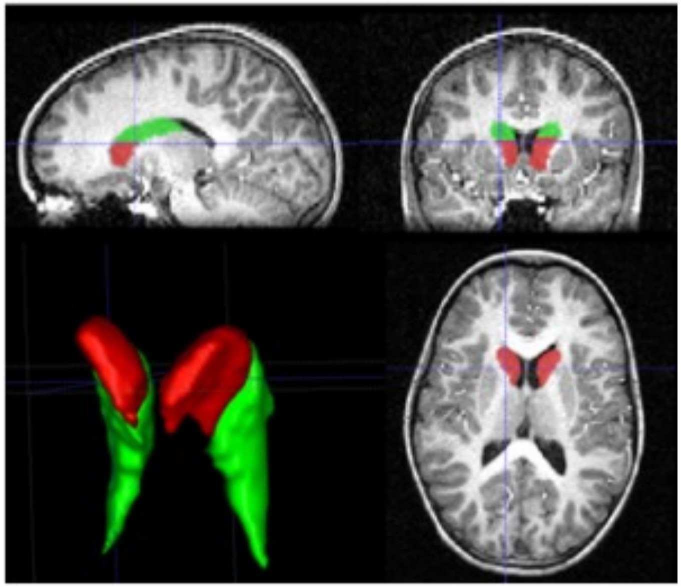


Scene understanding
[Fouhey et al., 2014; Ladicky et al., 2010;
Xiao et al., 2013; Yao et al., 2012]

MRFs / CRFs

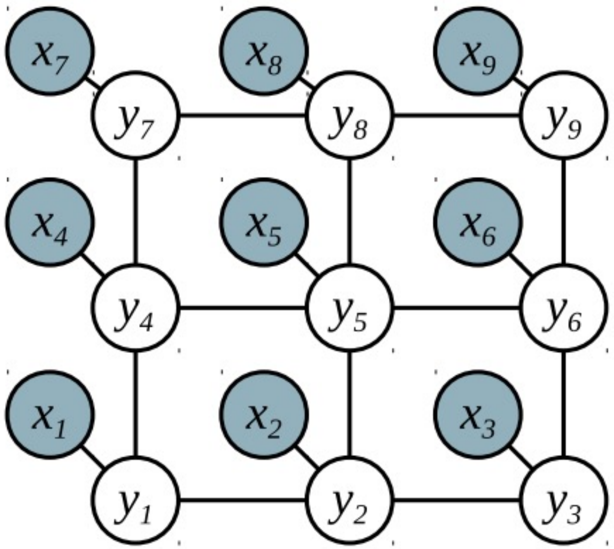
- Several applications, e.g., medical imaging



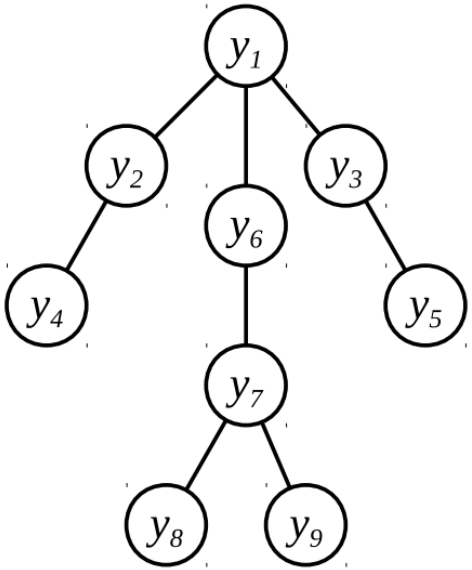


MRFs / CRFs

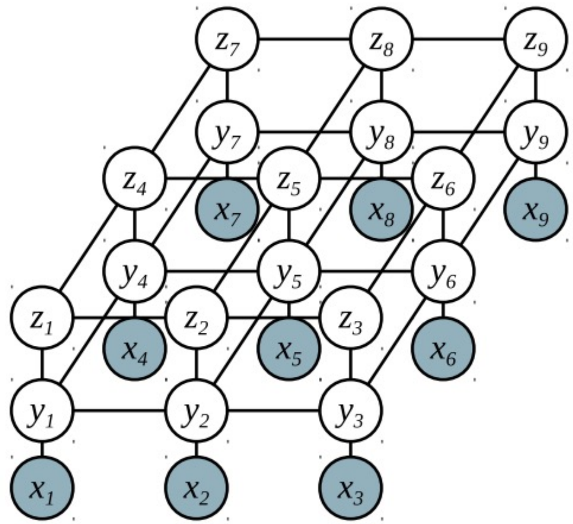
- Inherent in all these problems are graphical models



Pixel labeling



Object detection
Pose estimation



Scene understanding

Maximum a posteriori (MAP) inference

$$
\begin{aligned}
\mathbf{y}^\star &= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \log \left(\frac{1}{Z(\mathbf{X})} \prod_c \Psi_c(\mathbf{Y}_c; \mathbf{X}) \right) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_c \log \Psi_c(\mathbf{Y}_c; \mathbf{X}) - \log Z(\mathbf{X}) \\
&= \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_c \log \Psi_c(\mathbf{Y}_c; \mathbf{X})
\end{aligned}
$$

$$\log \Psi_c(\mathbf{Y}_c; \mathbf{X}) \rightarrow -E(\mathbf{Y}; \mathbf{X})$$

Maximum a posteriori (MAP) inference

$$\mathbf{y}^{\star} = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} P(\mathbf{y} \mid \mathbf{x}) = \operatorname*{argmax}_{\mathbf{y}\in\mathcal{Y}} \sum_{c} \log \Psi_c(\mathbf{Y}_c; \mathbf{X})$$

$$= \operatorname*{argmin}_{\mathbf{y}\in\mathcal{Y}} E(\mathbf{y}; \mathbf{x})$$

MAP inference ⇔ Energy minimization

The energy function is $E(\mathbf{Y}; \mathbf{X}) = \sum_{c} \psi_c(\mathbf{Y}_c; \mathbf{X})$

where $\psi_c(\cdot) = -\log \Psi_c(\cdot)$

Clique potential

Clique potentials

- Defines a mapping from an assignment of random variables to a real number

$$\psi_c : \mathcal{Y}_c \times \mathcal{X} \rightarrow \mathbb{R}$$

- Encodes a preference for assignments to the random variables (lower is better)

- Parameterized as $\psi_c(\mathbf{y}_c; \mathbf{x}) = \mathbf{w}_c^T \phi_c(\mathbf{y}_c; \mathbf{x})$

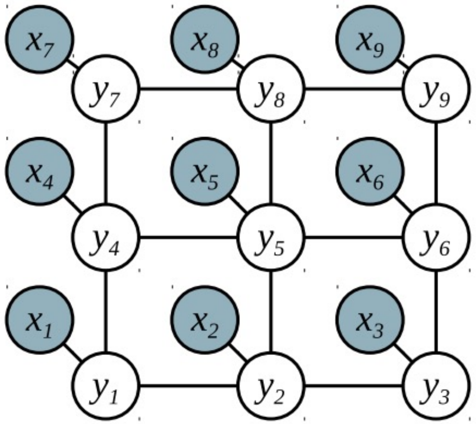
Parameters

Clique potentials

- Arity

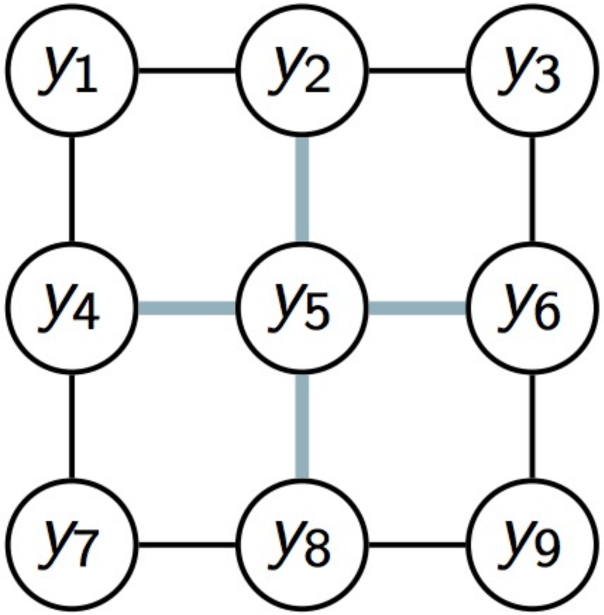
$$E(\mathbf{y};\mathbf{x}) = \sum_{c} \psi_c(\mathbf{y}_c;\mathbf{x})$$

$$= \underbrace{\sum_{i\in\mathcal{V}} \psi_i^U(y_i;\mathbf{x})}_{\text{unary}} + \underbrace{\sum_{ij\in\mathcal{E}} \psi_{ij}^P(y_i, y_j;\mathbf{x})}_{\text{pairwise}} + \underbrace{\sum_{c\in\mathcal{C}} \psi_c^H(\mathbf{y}_c;\mathbf{x})}_{\text{higher-order}}.$$

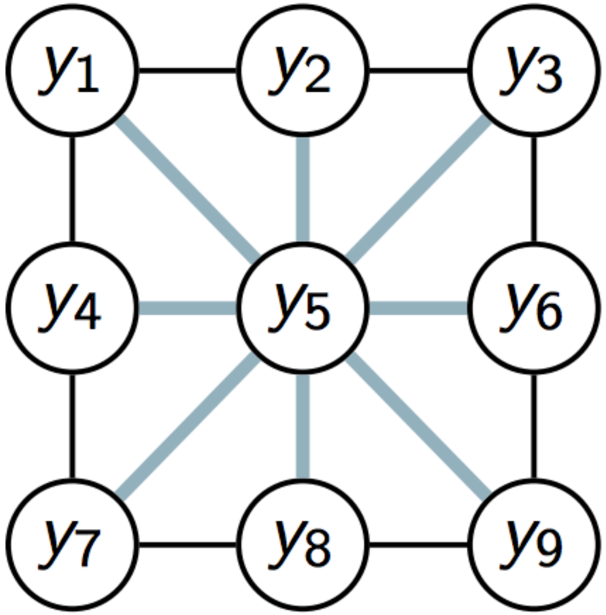


Clique potentials

- Arity

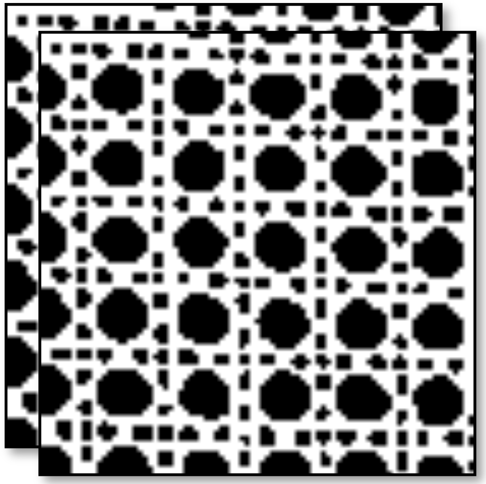


4-connected, $\mathcal{N}_4$

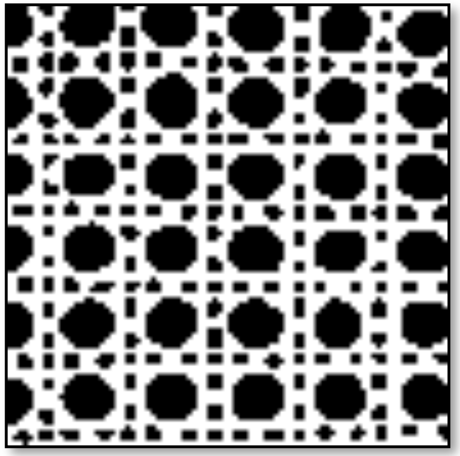


8-connected, $\mathcal{N}_8$

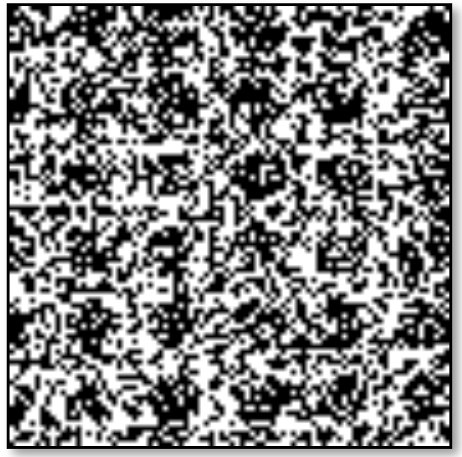
Reason 1: Texture modelling



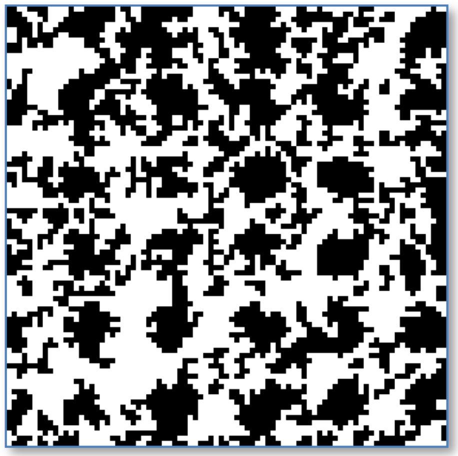
Training images



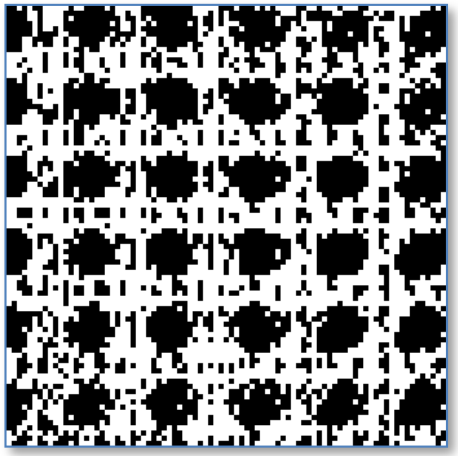
Test image



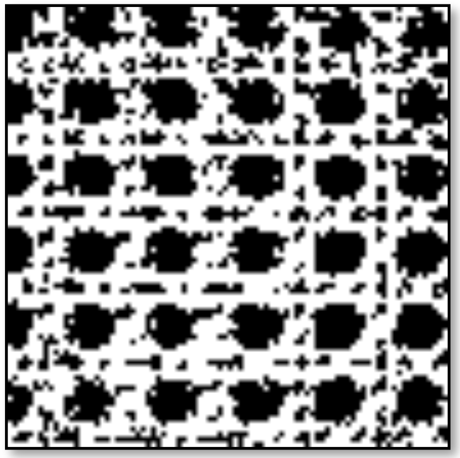
Test image (60% Noise)



Result MRF
4-connected
(neighbours)

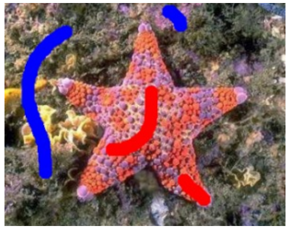


Result MRF
4-connected



Result MRF
9-connected
(7 attractive; 2 repulsive)

Reason2: Discretization artefacts



4-connected
Euclidean

8-connected
Euclidean

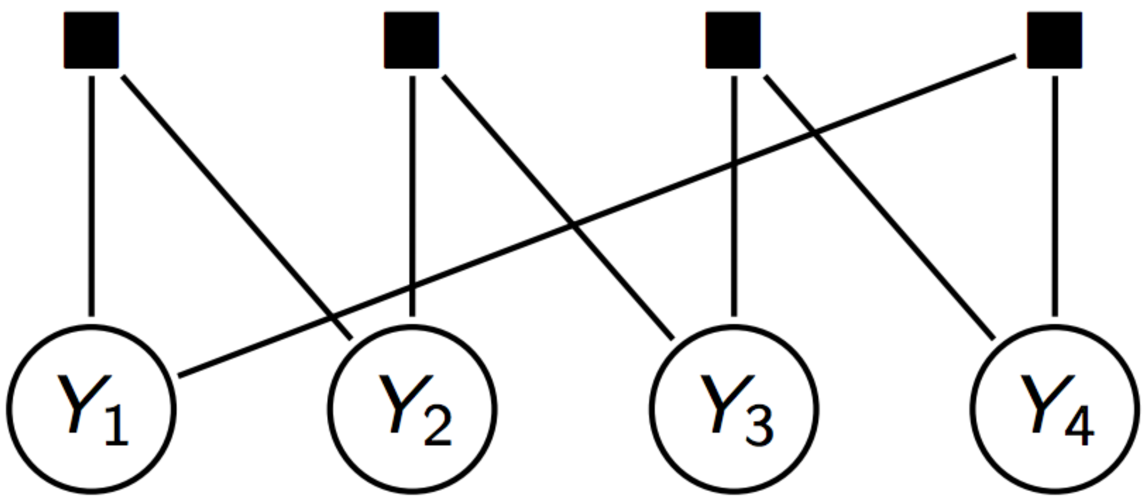
higher-connectivity can model true Euclidean length

[Boykov et al. ’03; ’05]

Graphical representation

- Example

$$E(\mathbf{y}) = \psi(y_1, y_2) + \psi(y_2, y_3) + \psi(y_3, y_4) + \psi(y_4, y_1)$$

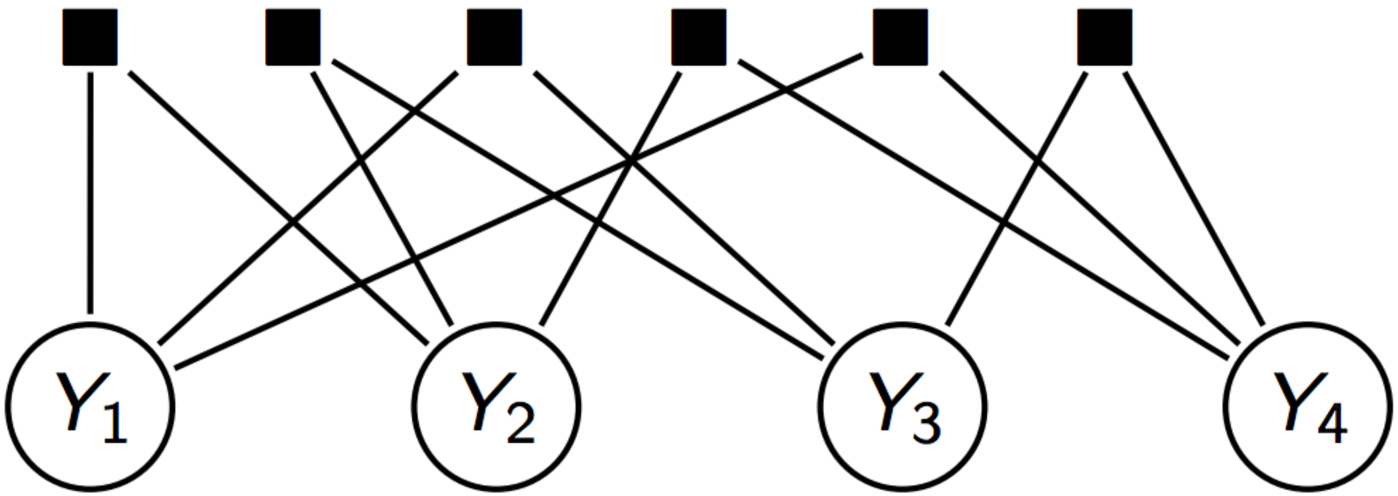


factor graph

Graphical representation

- Example

$$E(\mathbf{y}) = \sum_{i,j} \psi(y_i, y_j)$$

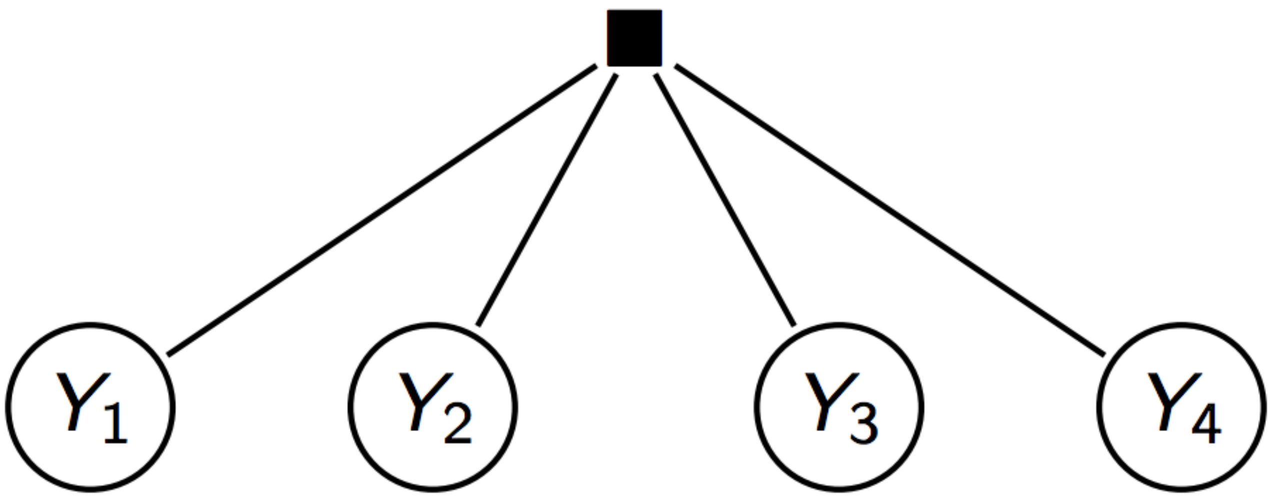


factor graph

Graphical representation

- Example

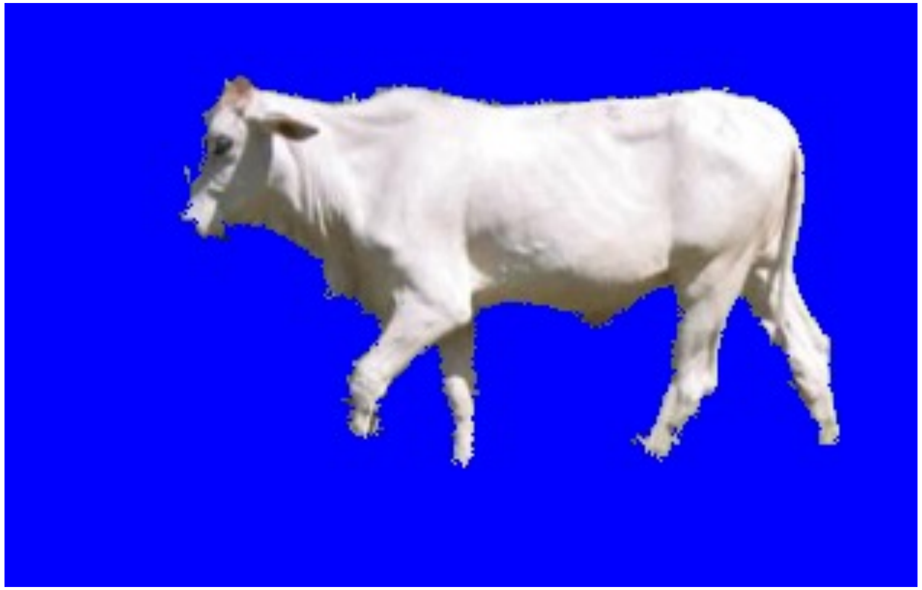
$$E(\mathbf{y}) = \psi(y_1, y_2, y_3, y_4)$$



factor graph

A Computer Vision Application

Binary Image Segmentation



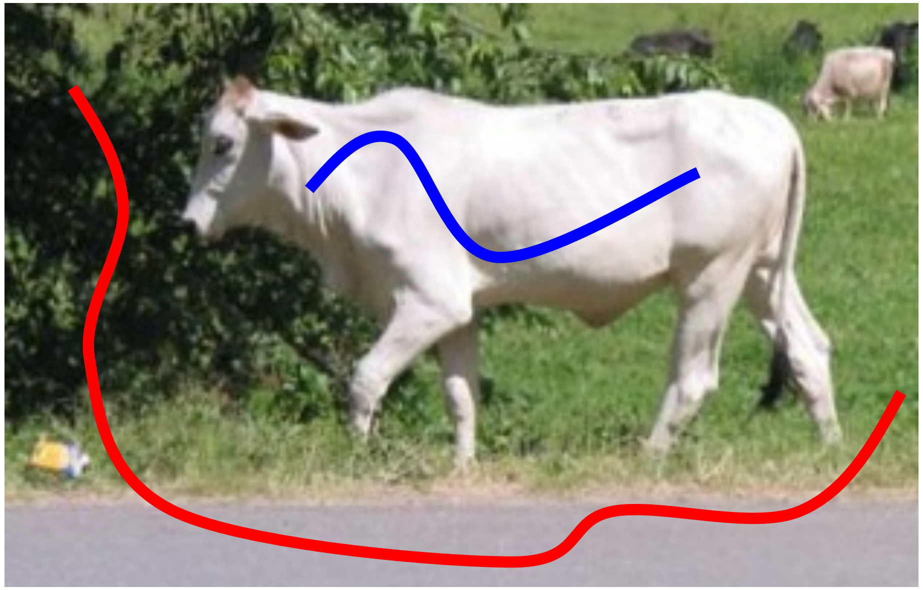
How ?

Cost function   Models *our* knowledge about natural images

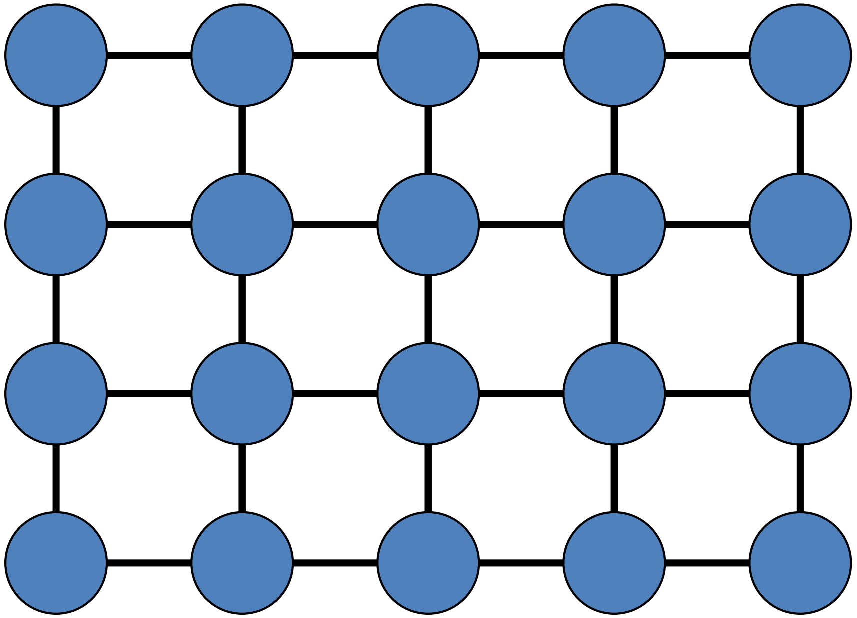
Optimize cost function to obtain the segmentation

A Computer Vision Application

Binary Image Segmentation



Object - white, Background - green/grey



Graph G = (V,E)

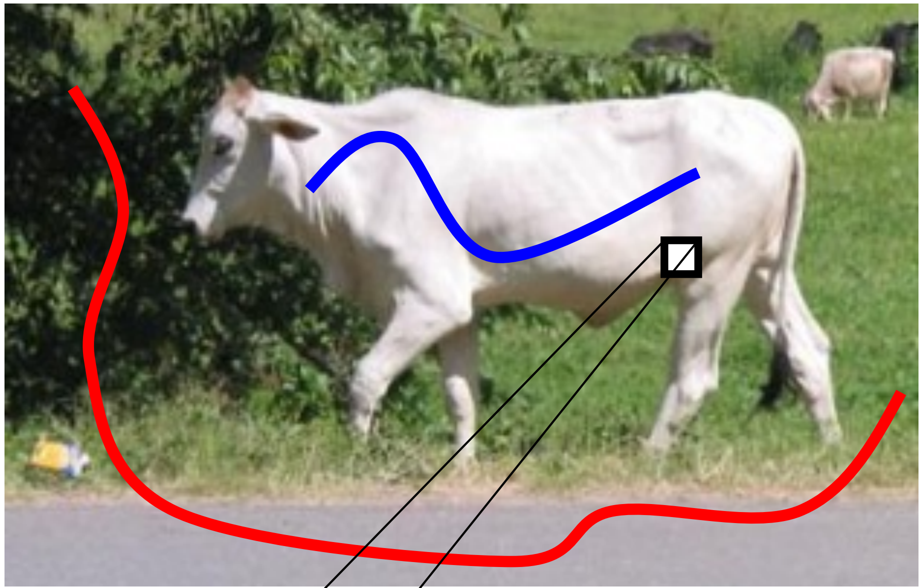
Each vertex corresponds to a pixel

Edges define a 4-neighbourhood *grid* graph

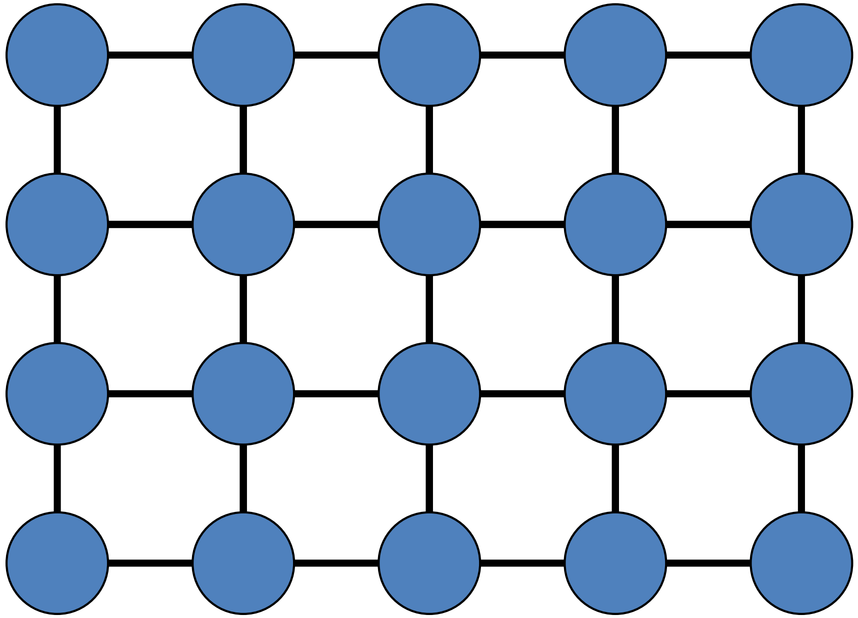
Assign a label to each vertex from L = {obj,bkg}

A Computer Vision Application

Binary Image Segmentation



Object - white, Background - green/grey



Graph G = (V,E)

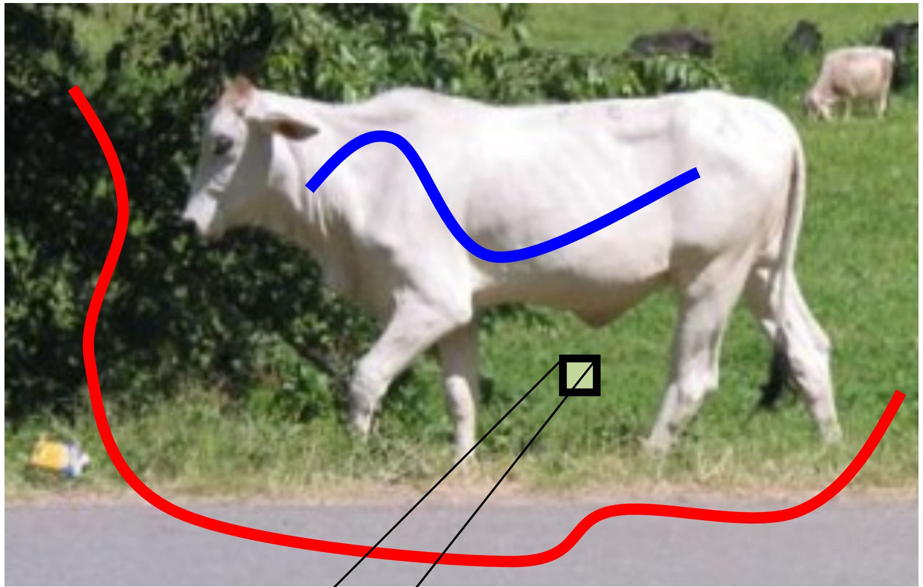
Cost of a labelling f : V ➔ L

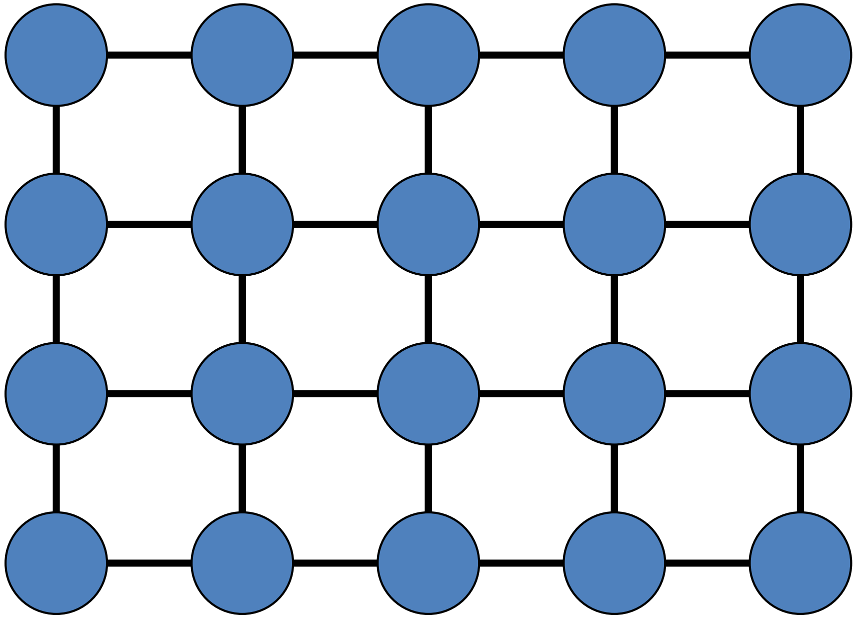
Per Vertex Cost

Cost of label 'obj' low Cost of label 'bkg' high

A Computer Vision Application

Binary Image Segmentation



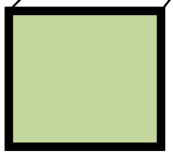


Object - white, Background - green/grey

Graph G = (V,E)

Cost of a labelling f : V ➔ L

Per Vertex Cost

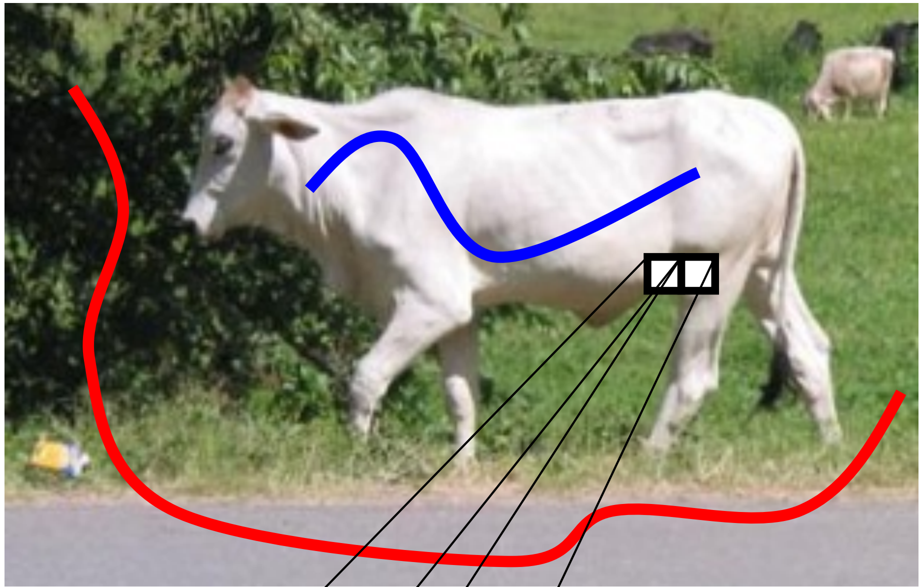


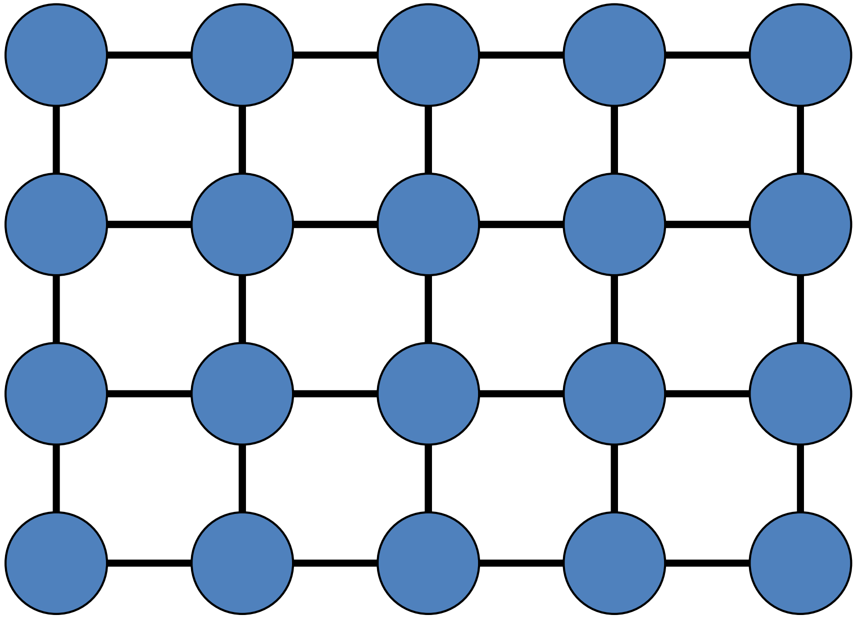
Cost of label ‘obj’ high Cost of label ‘bkg’ low

UNARY COST

A Computer Vision Application

Binary Image Segmentation



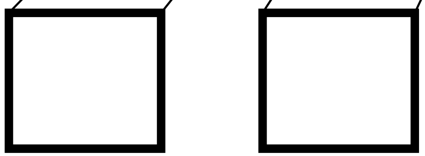


Object - white, Background - green/grey

Graph G = (V,E)

Cost of a labelling f : V ➔ L

Per Edge Cost

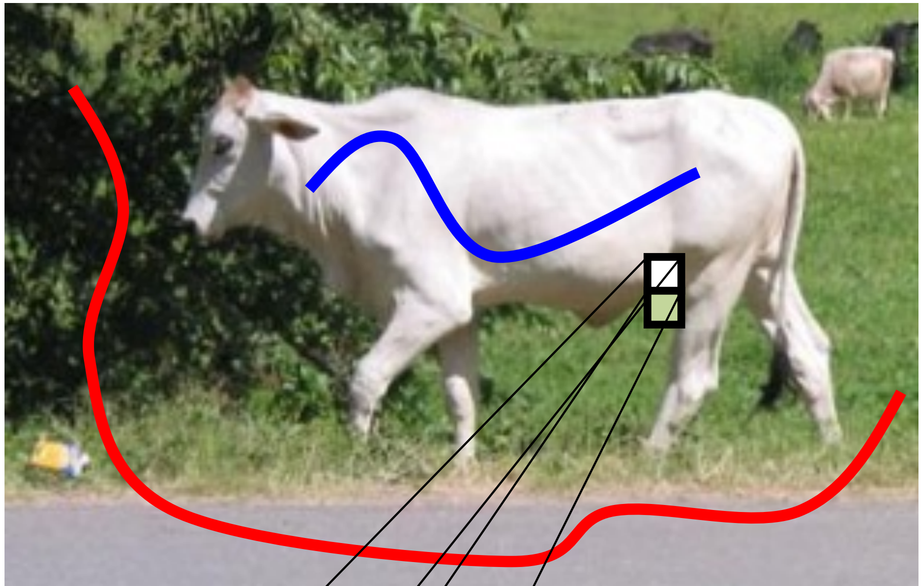


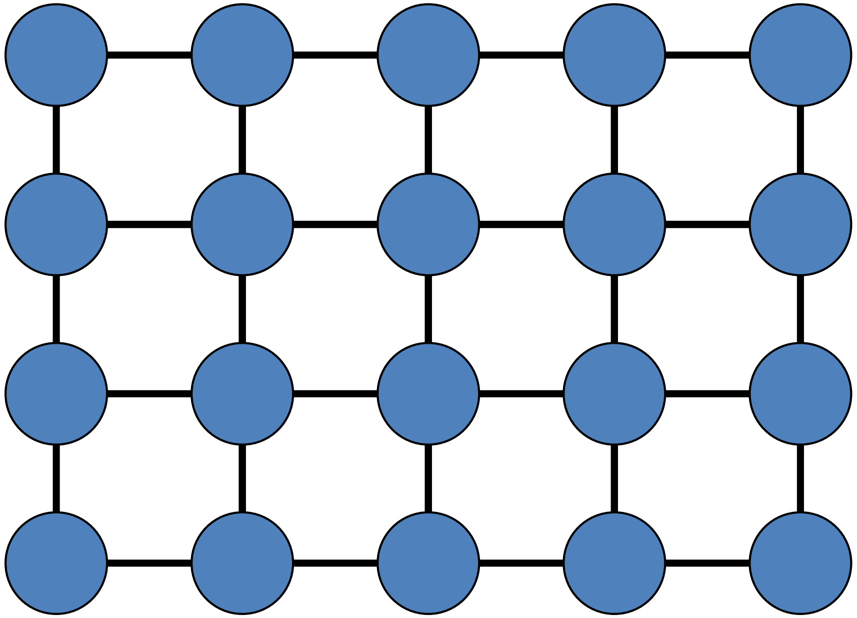
Cost of same label low

Cost of different labels high

A Computer Vision Application

Binary Image Segmentation



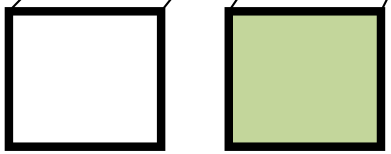


Object - white, Background - green/grey

Graph G = (V,E)

Cost of a labelling f : V ➔ L

Per Edge Cost



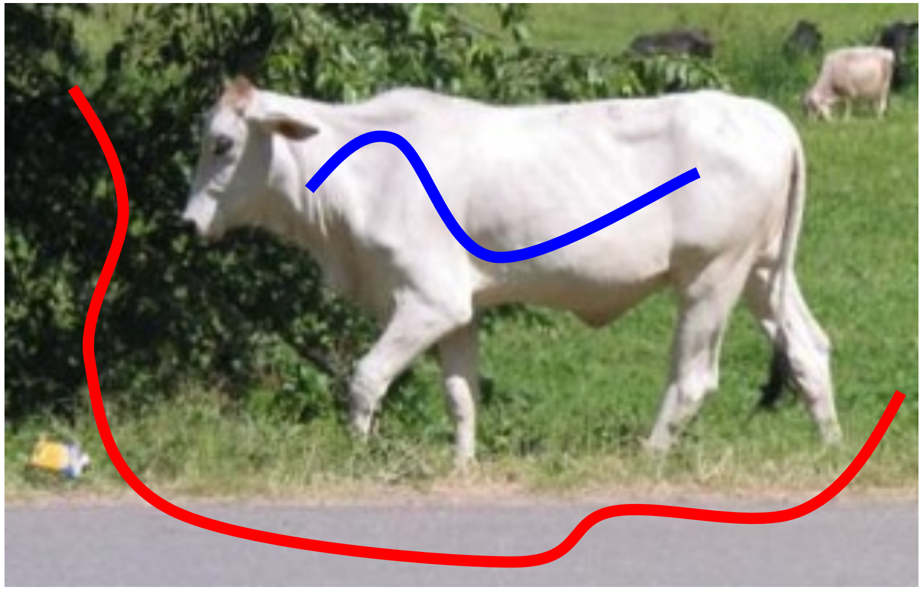
Cost of same label high

Cost of different labels low

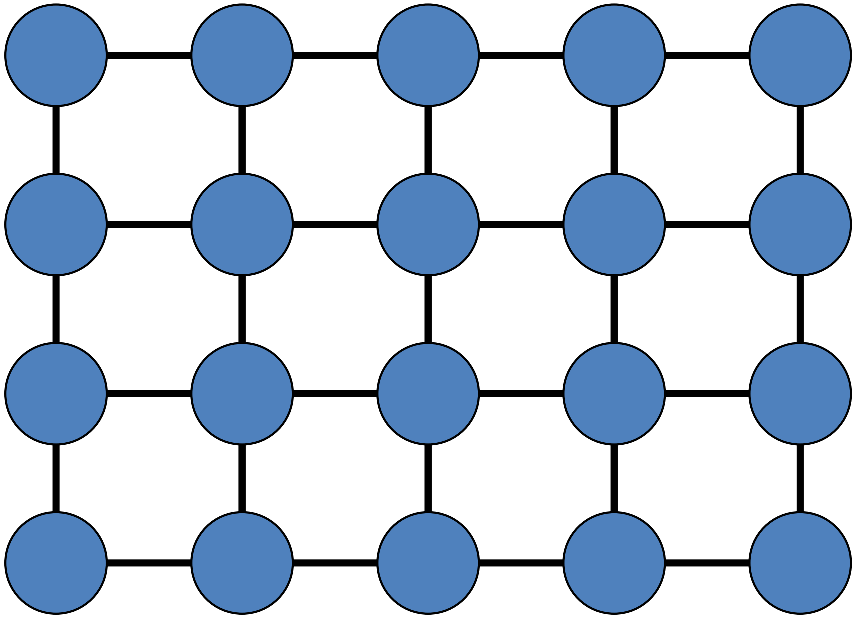
PAIRWISE COST

A Computer Vision Application

Binary Image Segmentation



Object - white, Background - green/grey

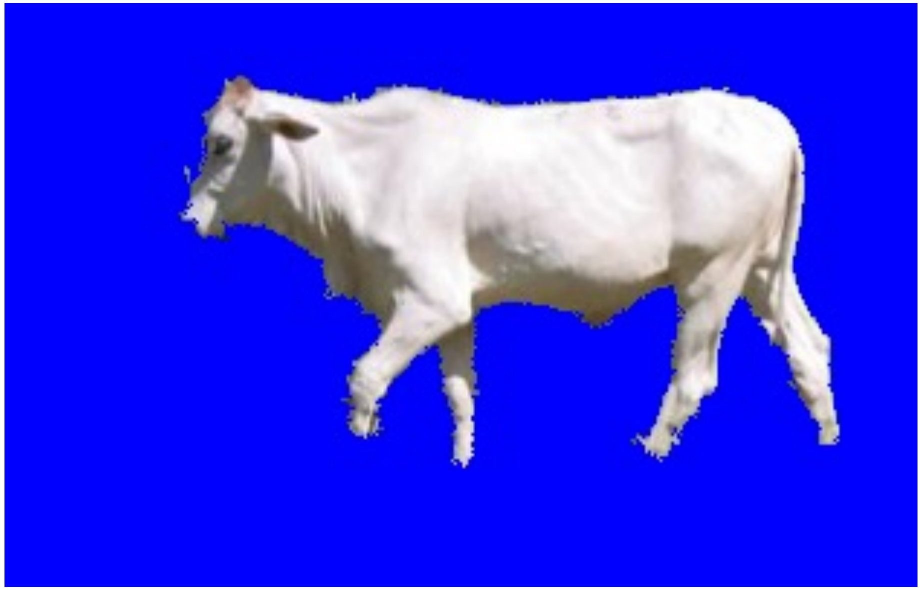


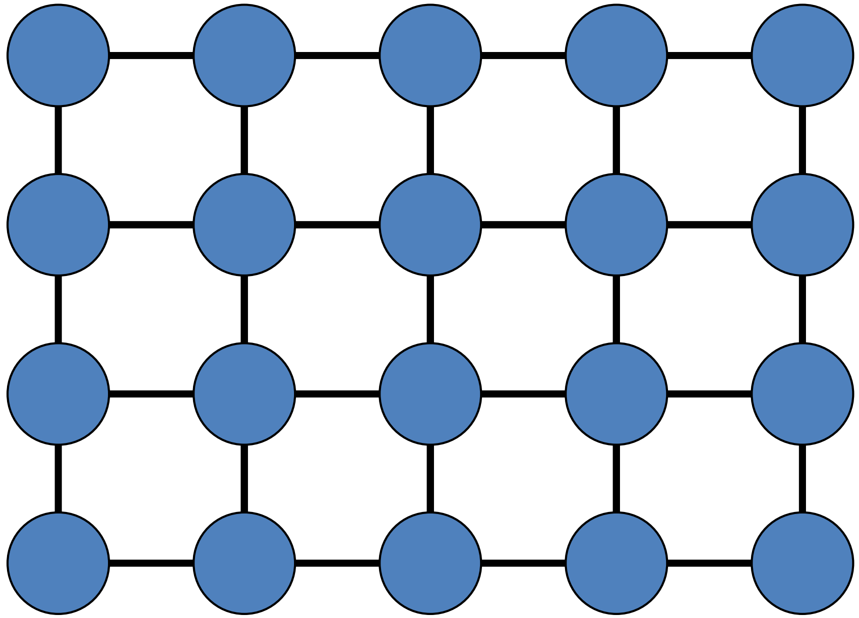
Graph G = (V,E)

Problem: Find the labelling with minimum cost f*

A Computer Vision Application

Binary Image Segmentation



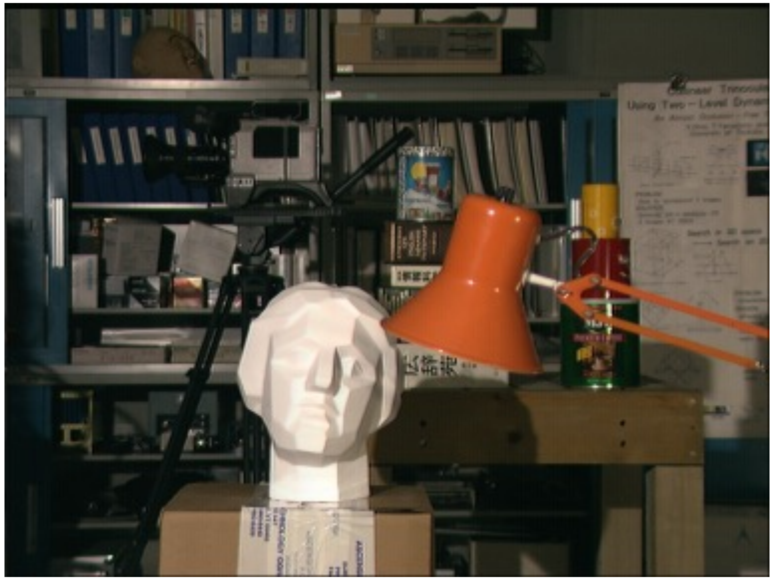


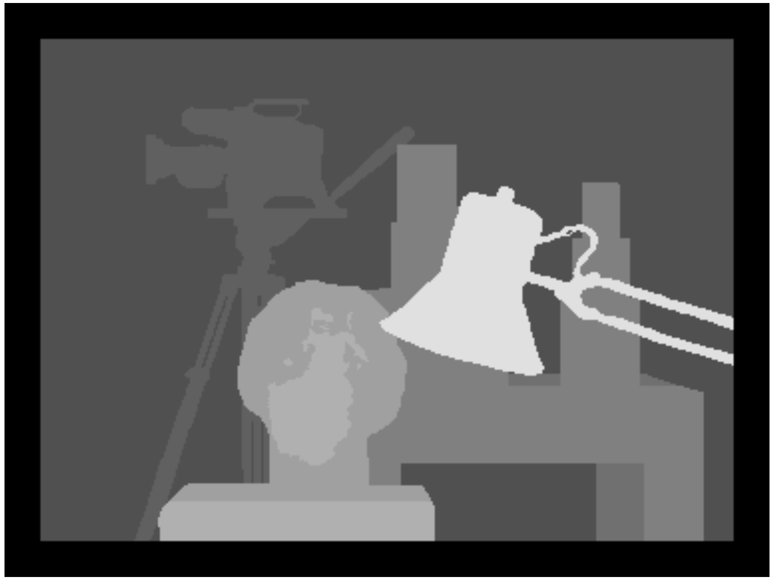
Graph G = (V,E)

Problem: Find the labelling with minimum cost f*

Another Computer Vision Application

Stereo Correspondence





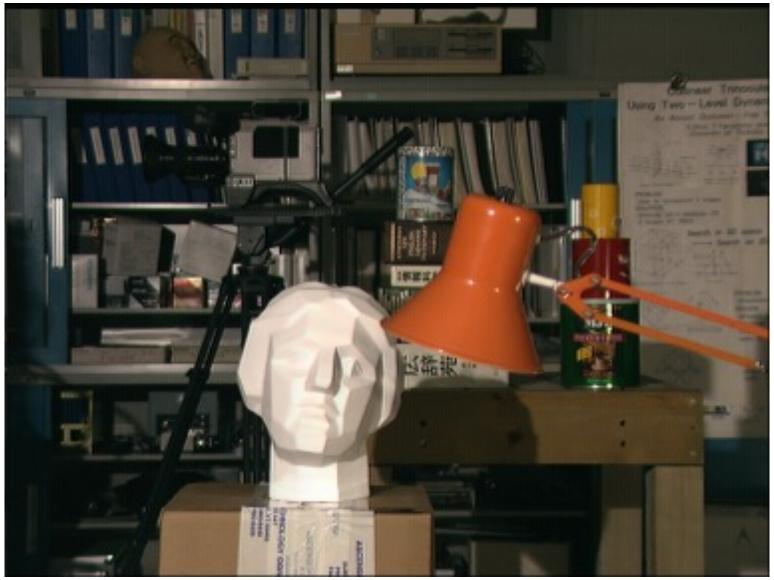
Disparity Map

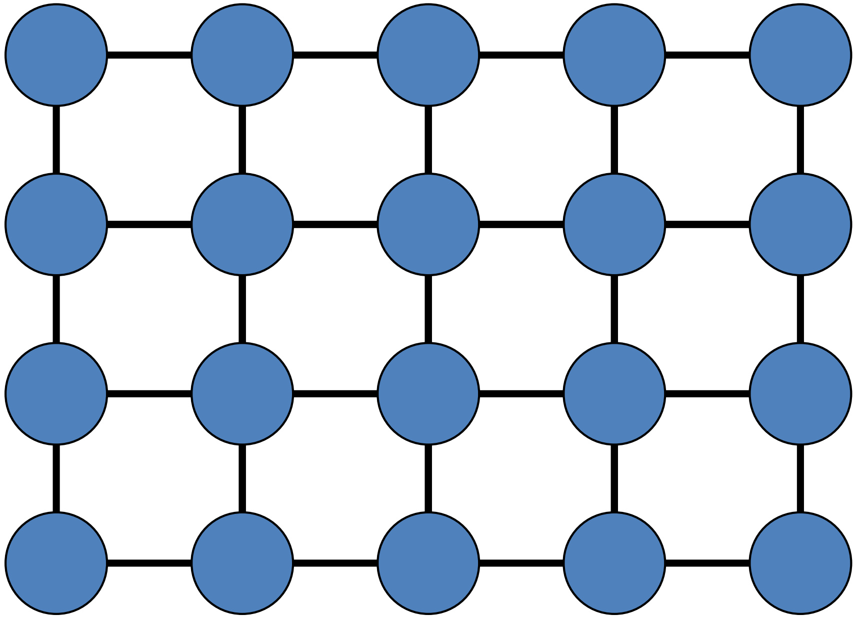
How ?

Minimizing a cost function

Another Computer Vision Application

Stereo Correspondence





Graph G = (V,E)

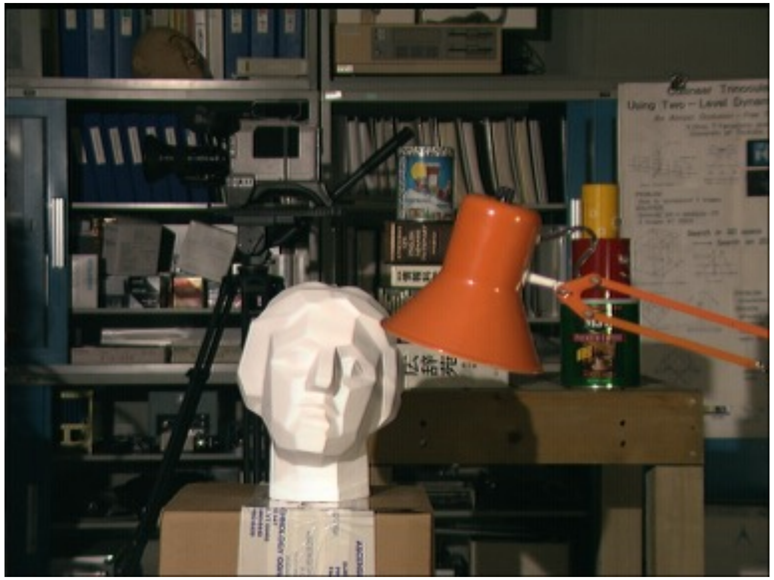
Vertex corresponds to a pixel

Edges define grid graph

L = {disparities}

Another Computer Vision Application

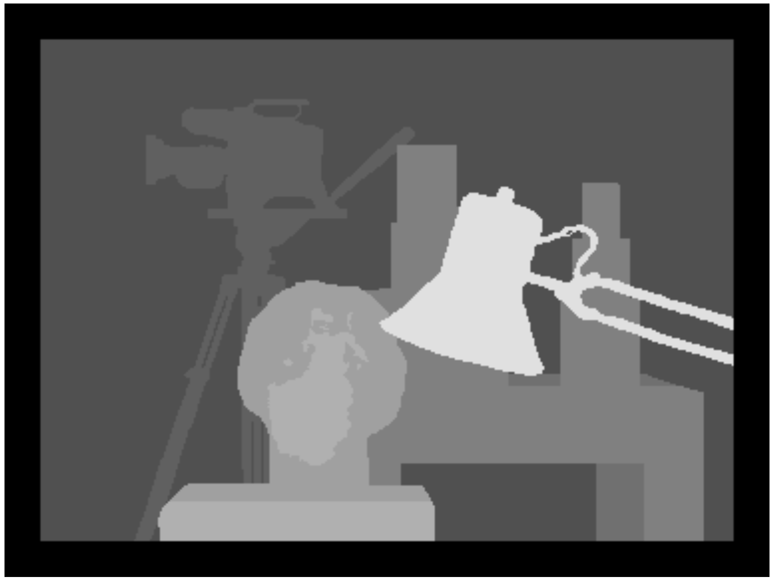
Stereo Correspondence



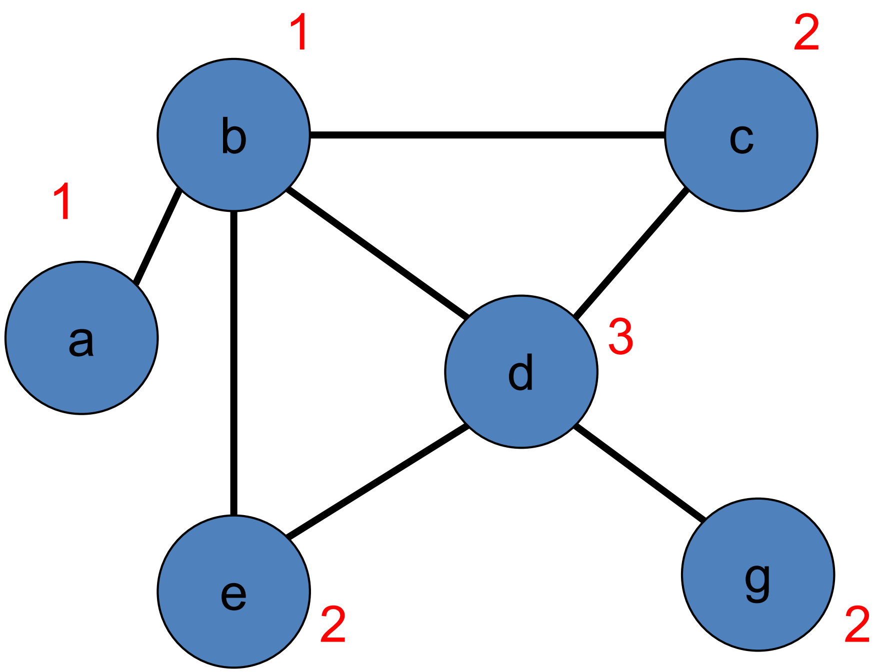
Cost of labelling f :

Unary cost + Pairwise Cost

Find minimum cost f*



The General Problem



Graph G = (V, E)

Discrete label set L = {1,2,…,h}

Assign a label to each vertex

f: V ➜ L

Cost of a labelling Q(f)

Unary Cost Pairwise Cost

Find f* = arg min Q(f)

Overview

- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
  - Belief Propagation and related methods
  - Graph cuts

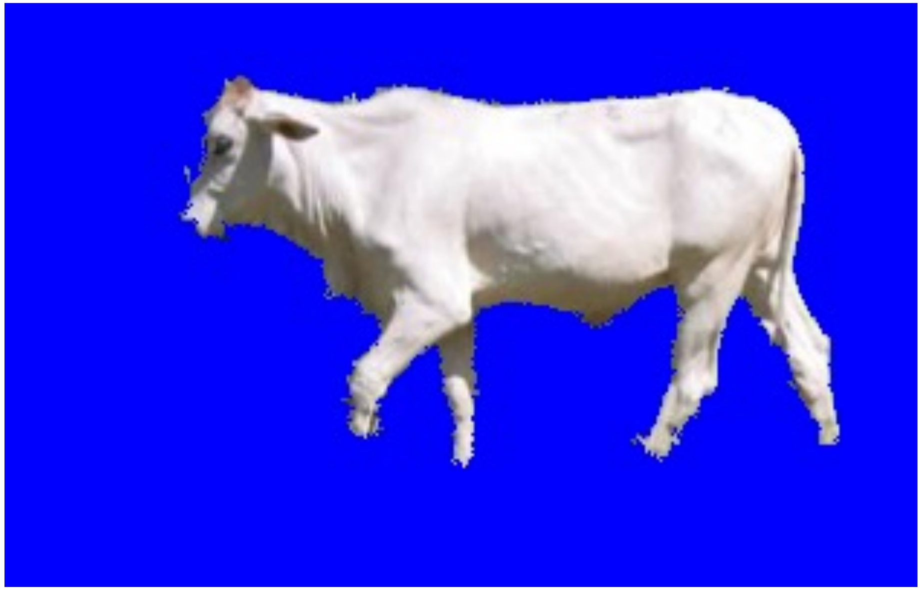
Remainder of today's lecture

- Belief propagation
- TRW
- Graph cuts

Belief Propagation

A Computer Vision Application

Binary Image Segmentation



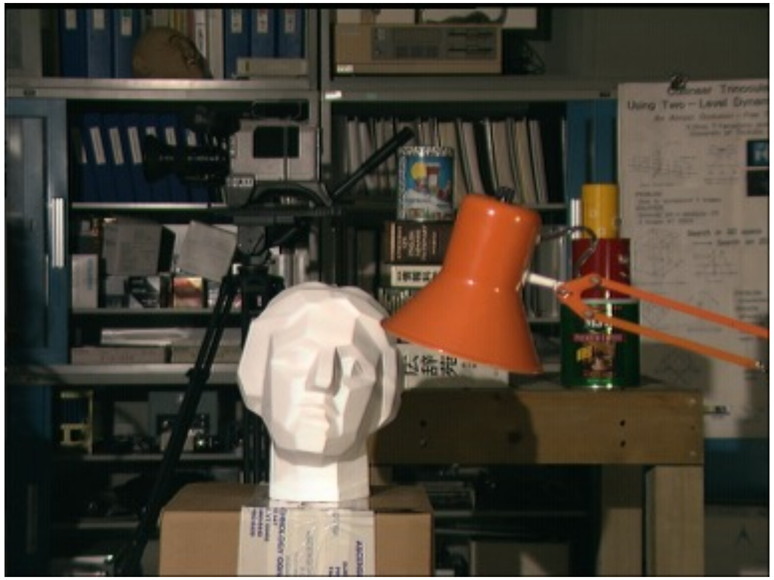
How ?

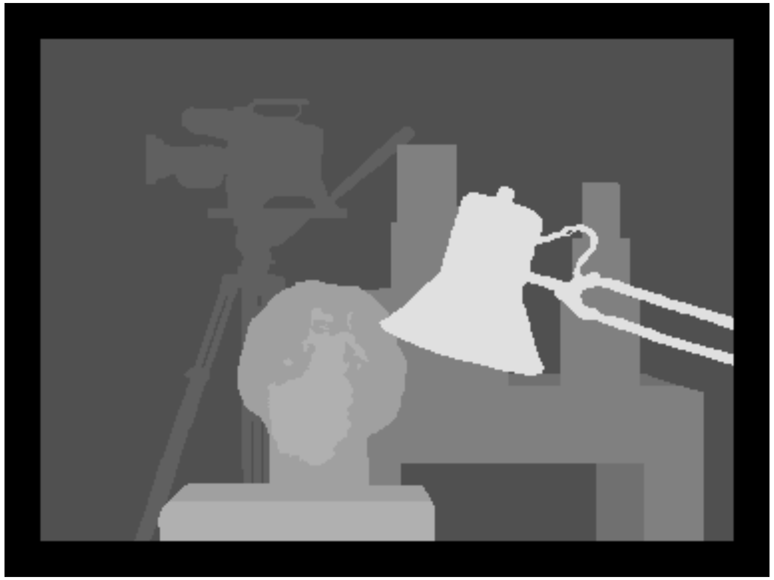
Cost function   Models *our* knowledge about natural images

Optimize cost function to obtain the segmentation

Another Computer Vision Application

Stereo Correspondence



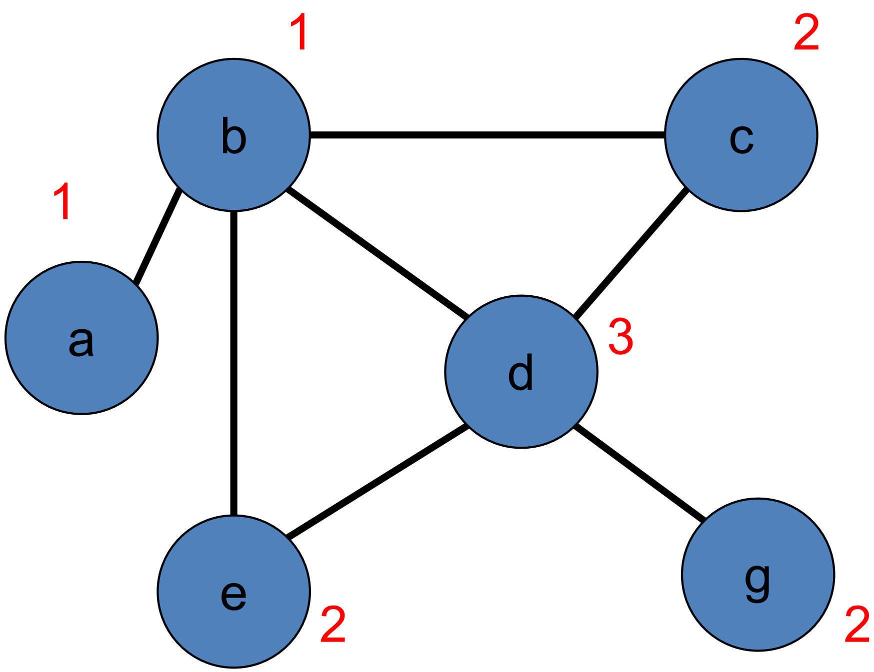


Disparity Map

How ?

Minimizing a cost function

The General Problem



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Discrete label set L = {1,2,…,h}

Assign a label to each vertex
f: V ➜ L

Cost of a labelling Q(f)

Unary Cost Pairwise Cost

Find f* = arg min Q(f)

Overview

- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
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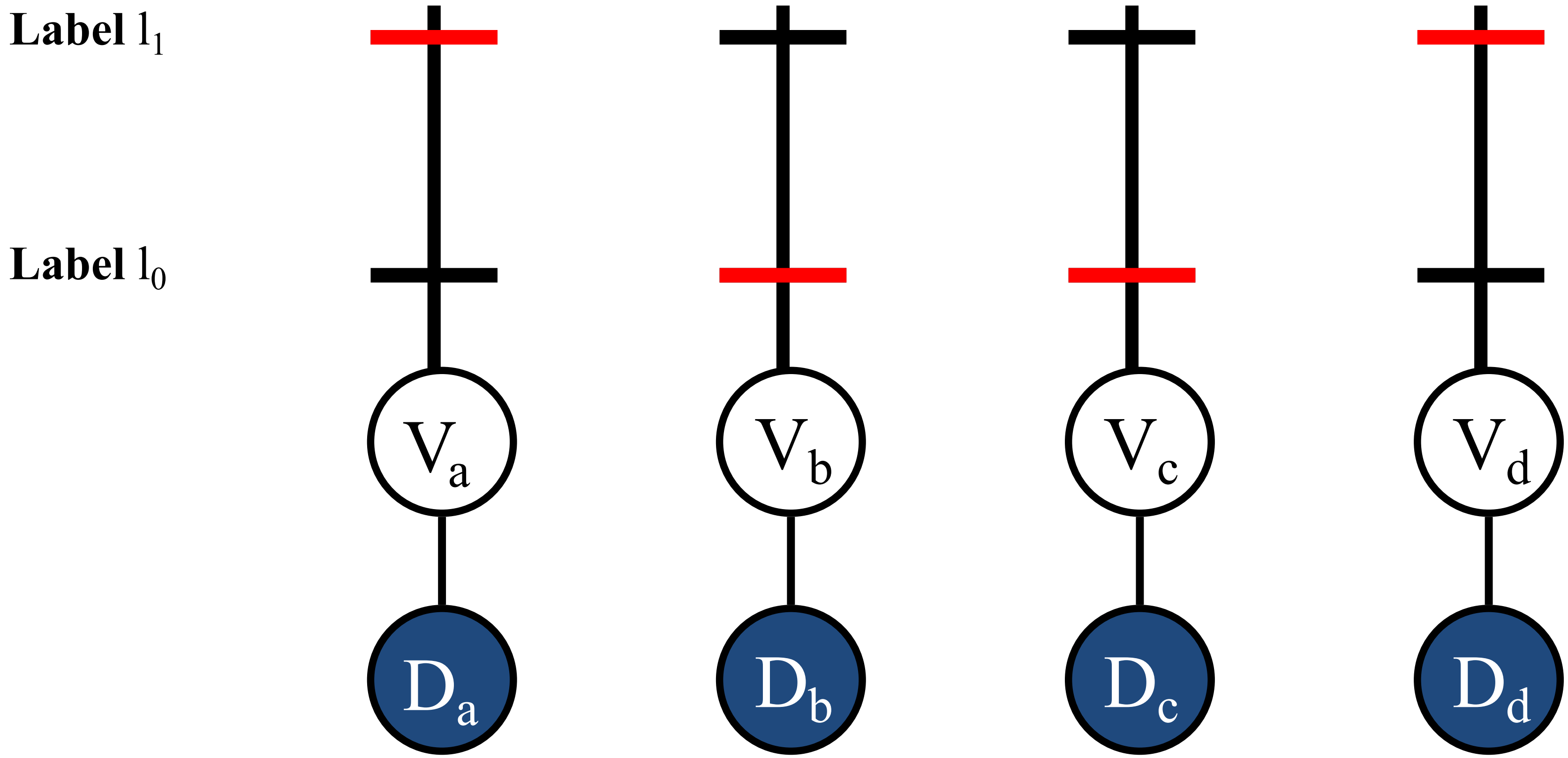
Energy Function

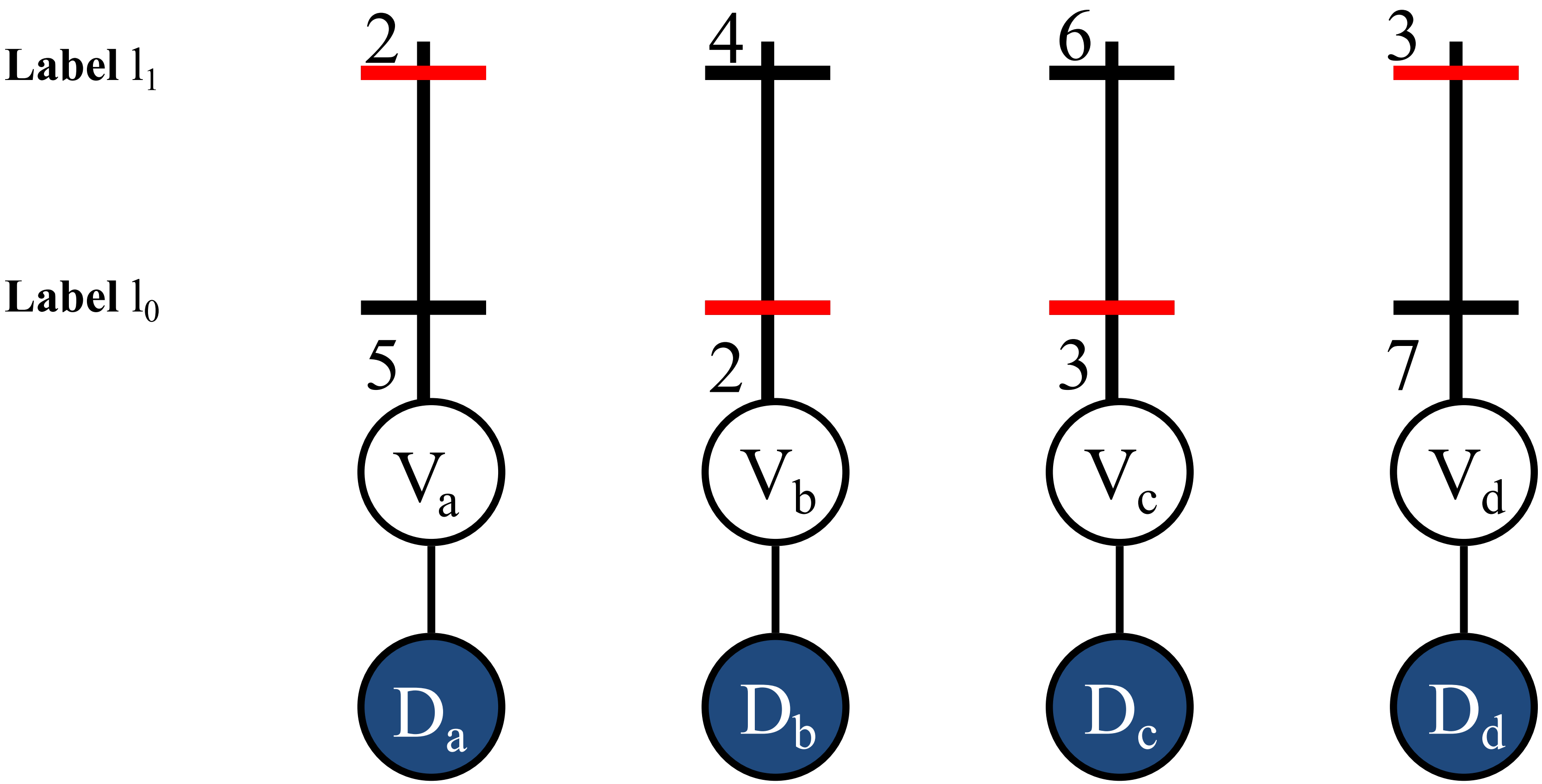


Random Variables $V = \{V_a, V_b, \ldots.\}$

Labels $L = \{l_0, l_1, \ldots.\}$ Data D

Labelling f: $\{a, b, \ldots. \} \rightarrow \{0,1, \ldots\}$

Energy Function



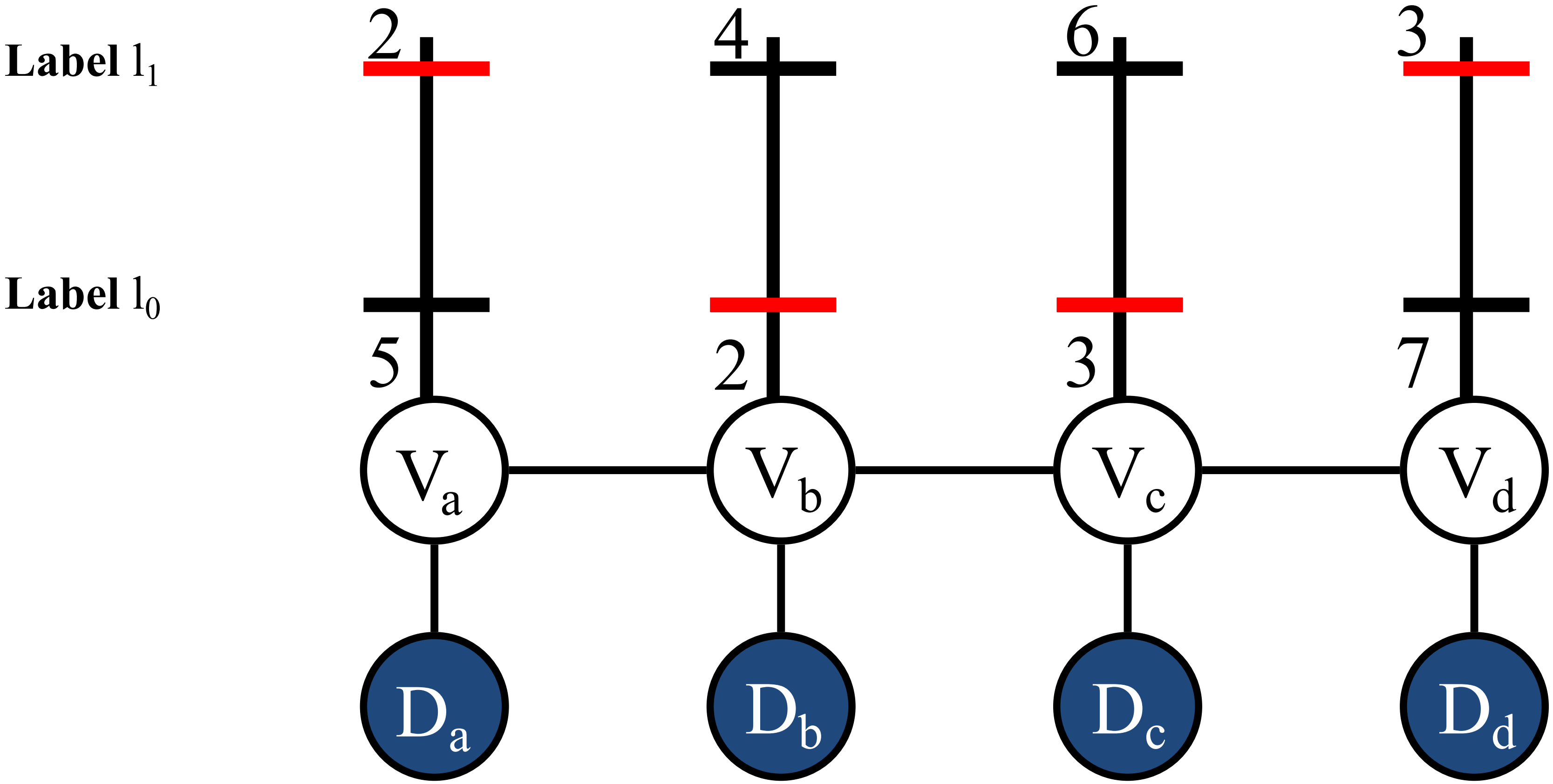
$$Q(f) = \sum_a \theta_{a;f(a)}$$

Unary Potential

Easy to minimize

Neighbourhood

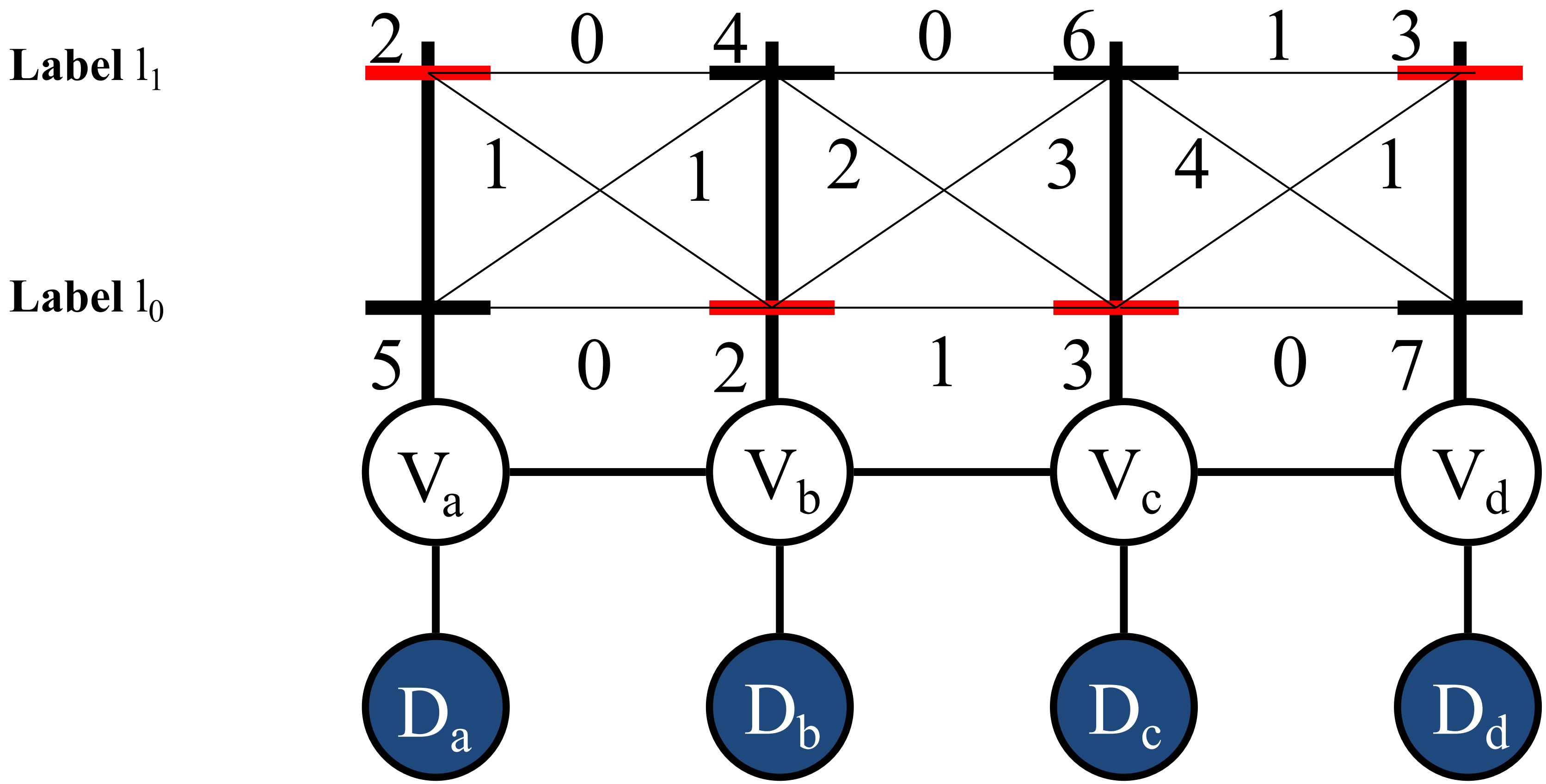
Energy Function



E : (a,b) $\in$ E iff V_a and V_b are neighbours

E = { (a,b) , (b,c) , (c,d) }

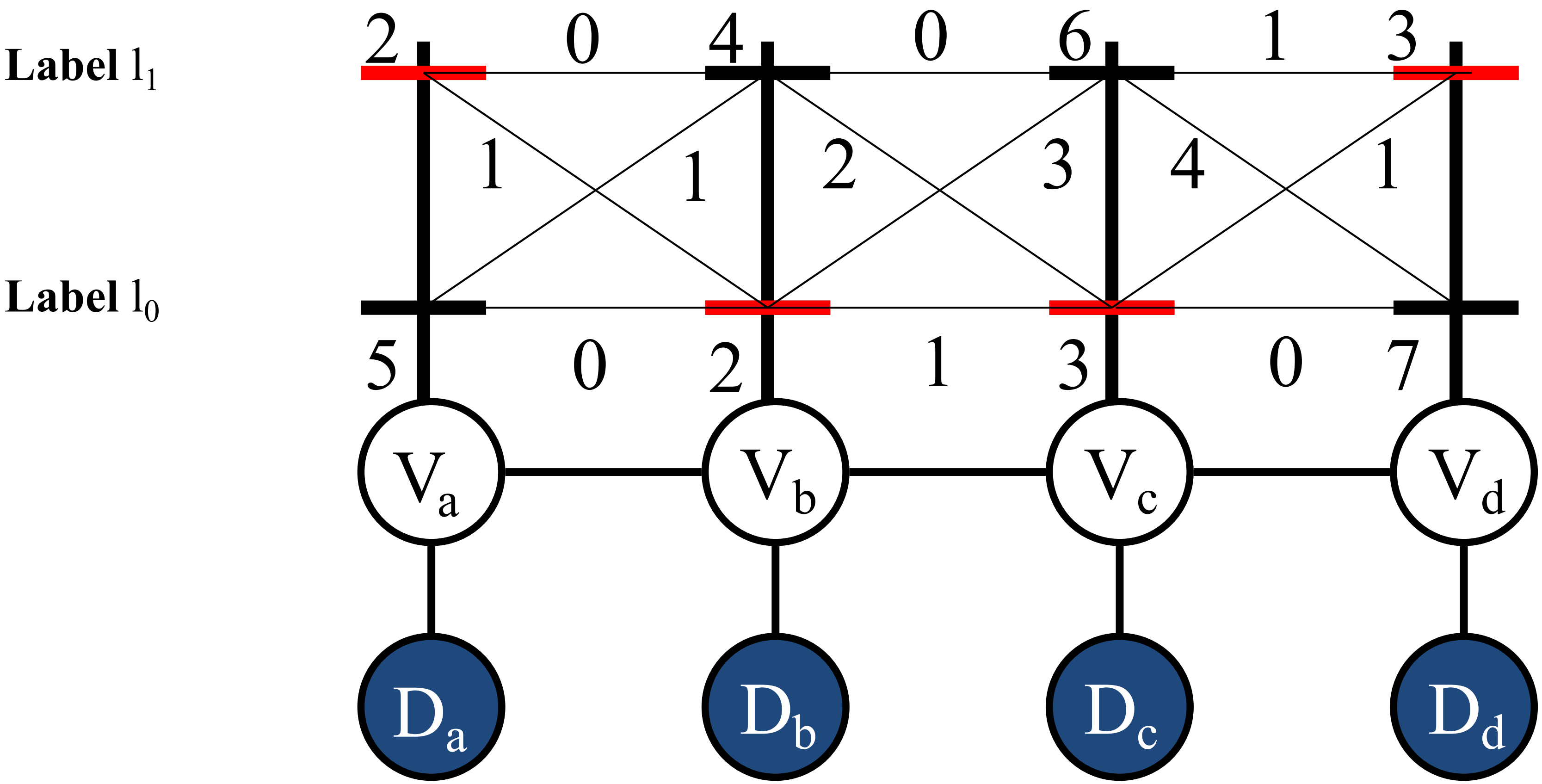
Energy Function



Pairwise Potential

$$Q(f) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

Energy Function



$$Q(f;\theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

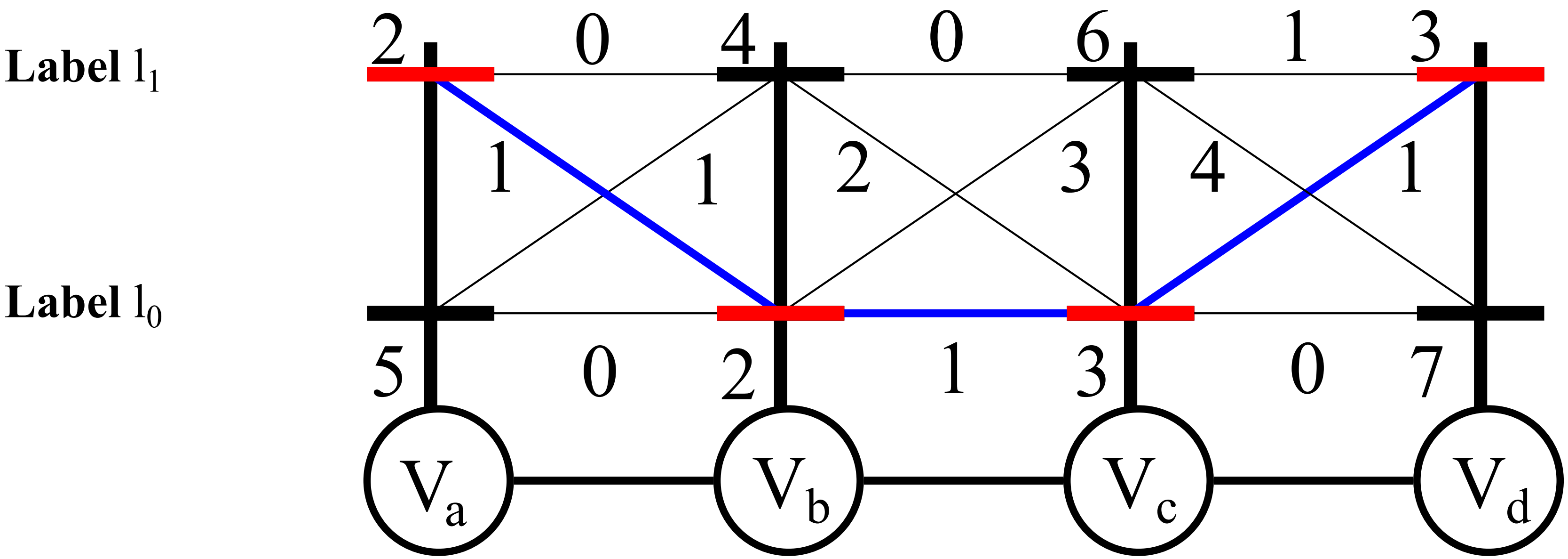
Parameter

Overview

- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
  - Belief Propagation and related methods
  - Graph cuts

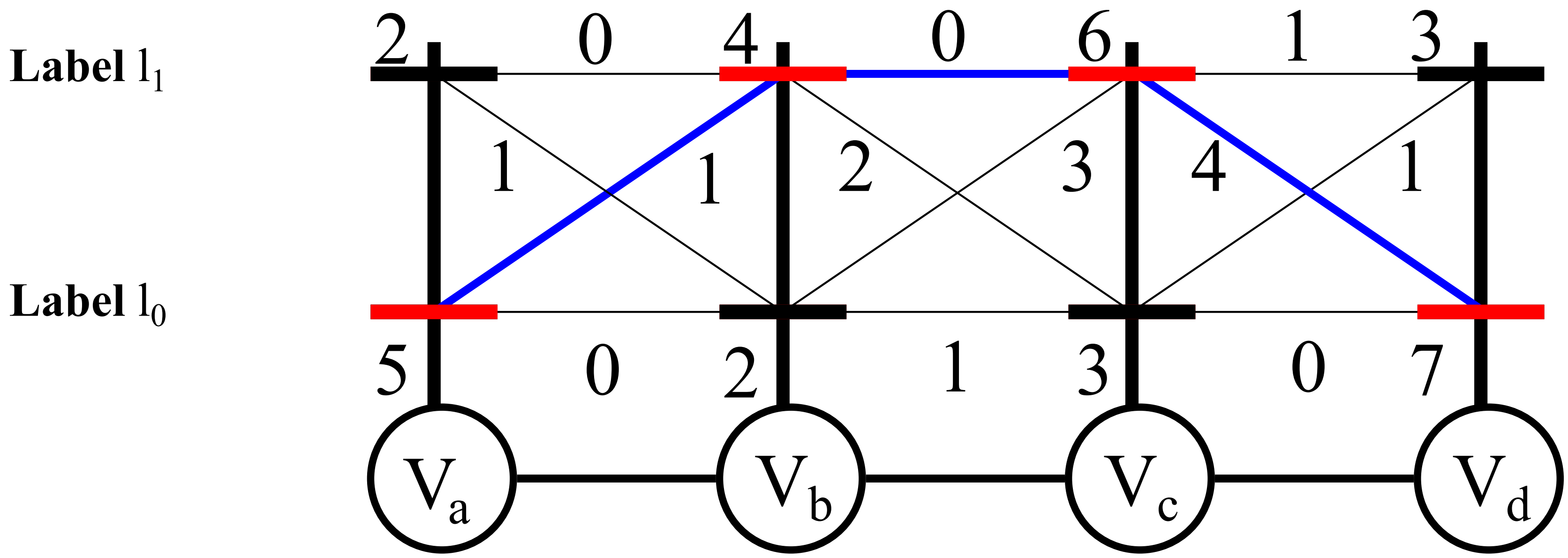
MAP Estimation



$$Q(f; \theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

2 + 1 + 2 + 1 + 3 + 1 + 3 = 13

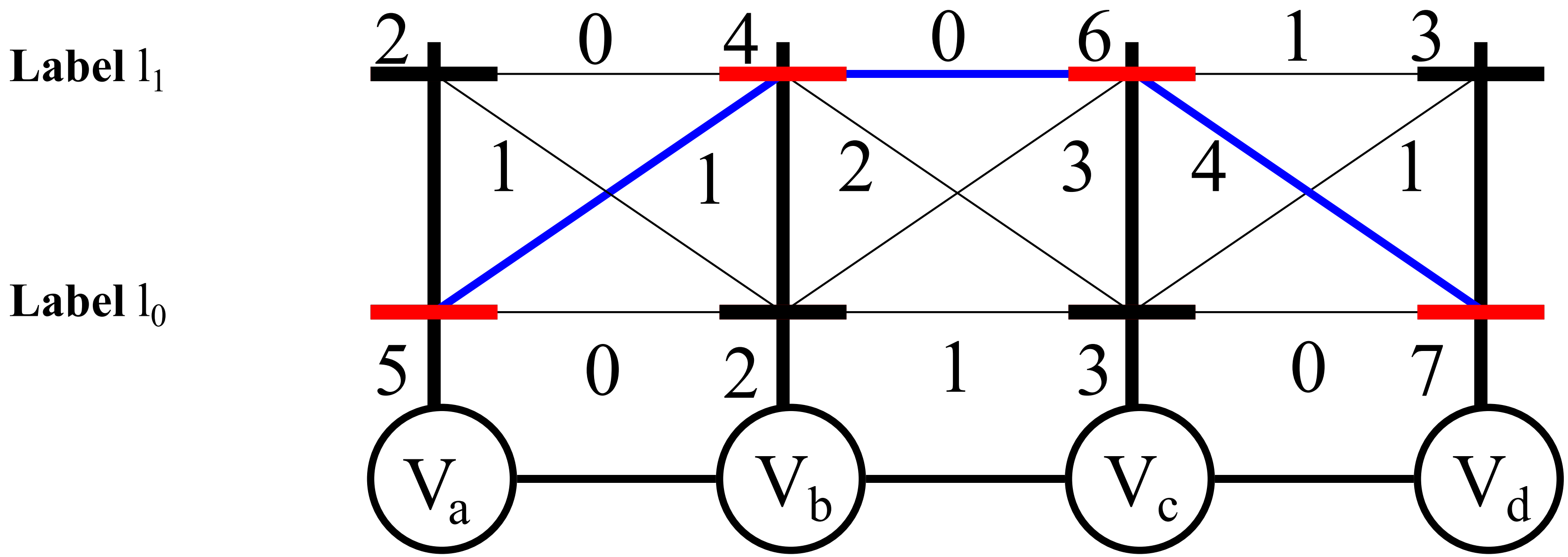
MAP Estimation



$$Q(f;\theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

$$5 + 1 + 4 + 0 + 6 + 4 + 7 = 27$$

MAP Estimation



$q^* = \min Q(f; \theta) = Q(f^*; \theta)$

$$Q(f; \theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

$f^* = \arg\min Q(f; \theta)$

Equivalent to maximizing the associated probability

MAP Estimation

16 possible labellings

$f^* = \{1, 0, 0, 1\}$

$q^* = 13$

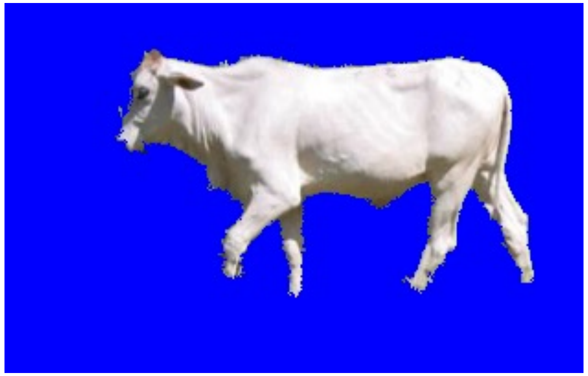
f(a)	f(b)	f(c)	f(d)	Q(f; θ)
0	0	0	0	18
0	0	0	1	15
0	0	1	0	27
0	0	1	1	20
0	1	0	0	22
0	1	0	1	19
0	1	1	0	27
0	1	1	1	20

f(a)	f(b)	f(c)	f(d)	Q(f; θ)
1	0	0	0	16
1	0	0	1	13
1	0	1	0	25
1	0	1	1	18
1	1	0	0	18
1	1	0	1	15
1	1	1	0	23
1	1	1	1	16

Computational Complexity

Segmentation

$2^{|V|}$



$|V|$ = number of pixels ≈ 153600

Can we do better than brute-force?

MAP Estimation is NP-hard !!

MAP Inference / Energy Minimization

- Computing the assignment minimizing the energy in NP-hard in general

$$\underset{\mathbf{y}\in\mathcal{Y}}{\operatorname{argmin}}\, E(\mathbf{y};\mathbf{x}) = \underset{\mathbf{y}\in\mathcal{Y}}{\operatorname{argmax}}\, P(\mathbf{y} \mid \mathbf{x})$$

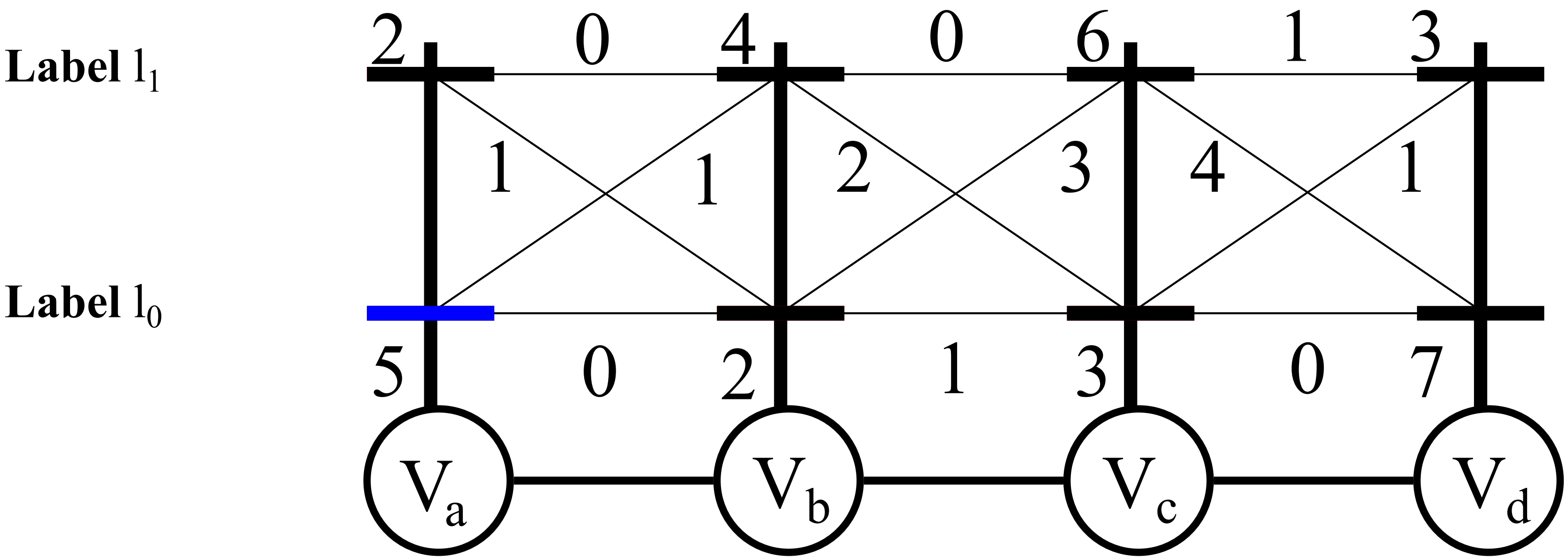
- Exact inference is possible in some cases, e.g.,
  - Low treewidth graphs → message-passing
  - Submodular potentials → graph cuts
- Efficient approximate inference algorithms exist
  - Message passing on general graphs
  - Move-making algorithms
  - Relaxation algorithms

Overview

- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
  - Belief Propagation and related methods
  - Graph cuts

Min-Marginals



Not a marginal (no summation)

$f^* = \arg\min Q(f; \theta)$ such that $f(a) = i$

Min-marginal $q_{a;i}$

Min-Marginals

16 possible labellings

$q_{a;0} = 15$

f(a)	f(b)	f(c)	f(d)	Q(f; θ)
0	0	0	0	18
0	0	0	1	15
0	0	1	0	27
0	0	1	1	20
0	1	0	0	22
0	1	0	1	19
0	1	1	0	27
0	1	1	1	20

f(a)	f(b)	f(c)	f(d)	Q(f; θ)
1	0	0	0	16
1	0	0	1	13
1	0	1	0	25
1	0	1	1	18
1	1	0	0	18
1	1	0	1	15
1	1	1	0	23
1	1	1	1	16

Min-Marginals

16 possible labellings

$q_{a;1} = 13$

f(a)	f(b)	f(c)	f(d)	Q(f; θ)
0	0	0	0	18
0	0	0	1	15
0	0	1	0	27
0	0	1	1	20
0	1	0	0	22
0	1	0	1	19
0	1	1	0	27
0	1	1	1	20

f(a)	f(b)	f(c)	f(d)	Q(f; θ)
1	0	0	0	16
1	0	0	1	13
1	0	1	0	25
1	0	1	1	18
1	1	0	0	18
1	1	0	1	15
1	1	1	0	23
1	1	1	1	16

Min-Marginals and MAP

- Minimum min-marginal of any variable = energy of MAP labelling

$$\min_i \; q_{a;i}$$

$$\min_i \left(\min_f Q(f; \theta) \quad \text{such that } f(a) = i \right)$$

V_a has to take one label

$$\min_f Q(f; \theta)$$

Summary

Energy Function

$$Q(f;\theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

MAP Estimation

$$f^* = \arg\min Q(f;\theta)$$

Min-marginals

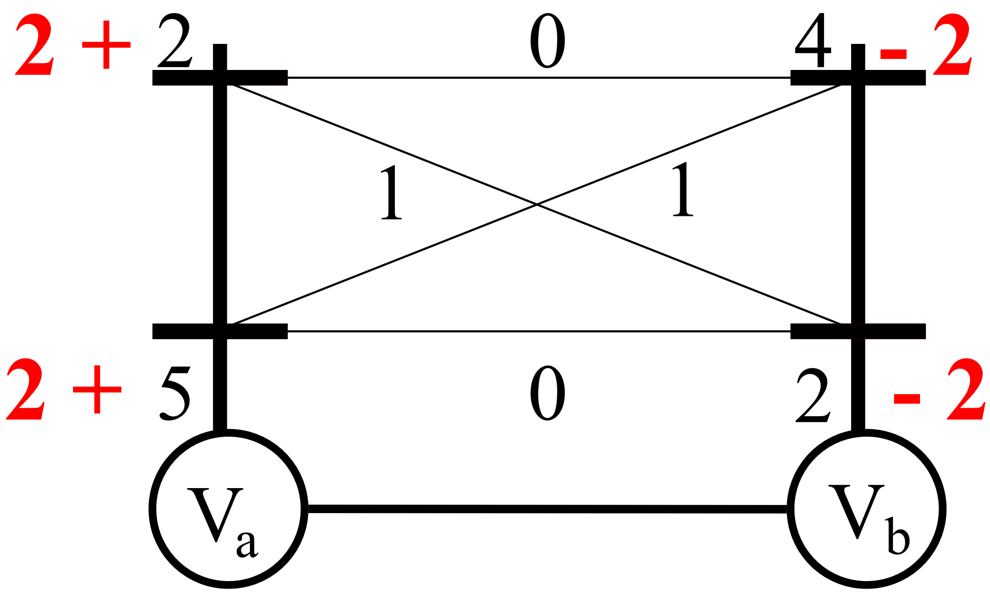
$$q_{a;i} = \min Q(f;\theta) \quad \text{s.t. } f(a) = i$$

Overview

- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
  - Belief Propagation and related methods
  - Graph cuts

Reparameterization



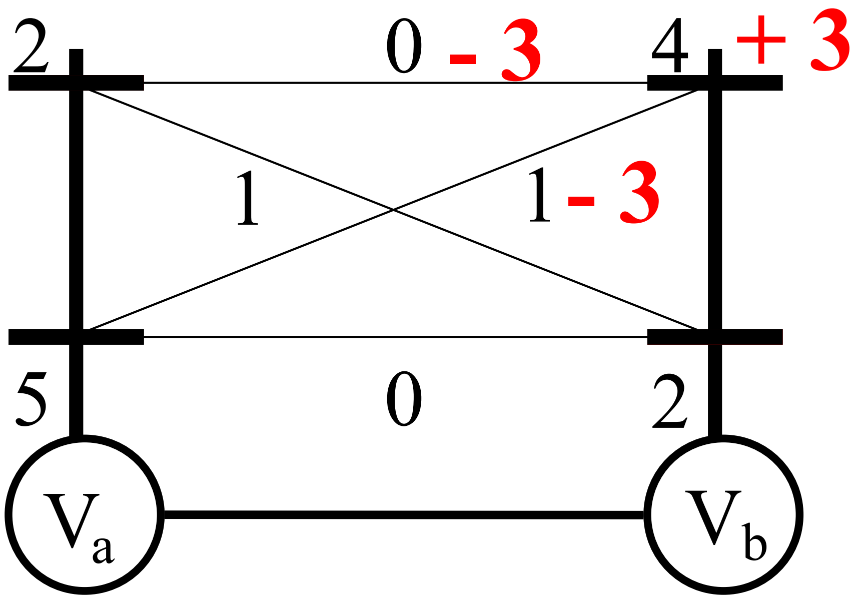
f(a)	f(b)	Q(f; θ)
0	0	7 + 2 - 2
0	1	10 + 2 - 2
1	0	5 + 2 - 2
1	1	6 + 2 - 2

Add a constant to all $\theta_{a;i}$

Subtract that constant from all $\theta_{b;k}$

$Q(f; \theta') = Q(f; \theta)$

Reparameterization



f(a)	f(b)	Q(f; θ)
0	0	7
0	1	10 **- 3 + 3**
1	0	5
1	1	6 **- 3 + 3**

Add a constant to one $\theta_{b;k}$

Subtract that constant from $\theta_{ab;ik}$ for all 'i'

$Q(f; \theta') = Q(f; \theta)$

Reparameterization

θ' is a reparameterization of θ, iff

$$Q(f; \theta') = Q(f; \theta), \text{ for all } f \qquad \theta' \equiv \theta$$

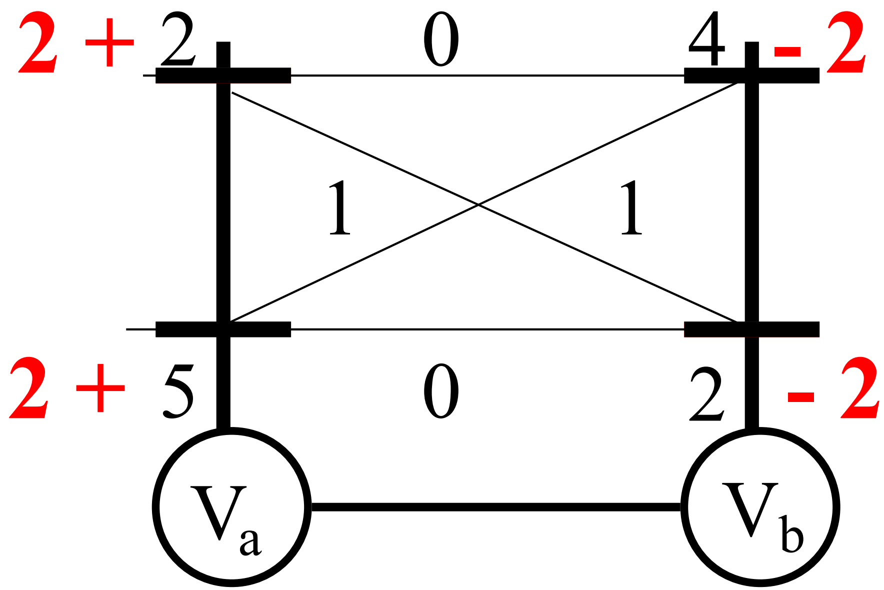
Equivalently

Kolmogorov, PAMI, 2006

$$\theta'_{a;i} = \theta_{a;i} + M_{ba;i}$$

$$\theta'_{b;k} = \theta_{b;k} + M_{ab;k}$$

$$\theta'_{ab;ik} = \theta_{ab;ik} - M_{ab;k} - M_{ba;i}$$



Recap

MAP Estimation

$$f^* = \arg\min Q(f; \theta)$$

$$Q(f; \theta) = \sum_a \theta_{a;f(a)} + \sum_{(a,b)} \theta_{ab;f(a)f(b)}$$

Min-marginals

$$q_{a;i} = \min Q(f; \theta) \quad \text{s.t. } f(a) = i$$

Reparameterization

$$Q(f; \theta') = Q(f; \theta), \text{ for all } f \qquad \theta' \equiv \theta$$

Overview

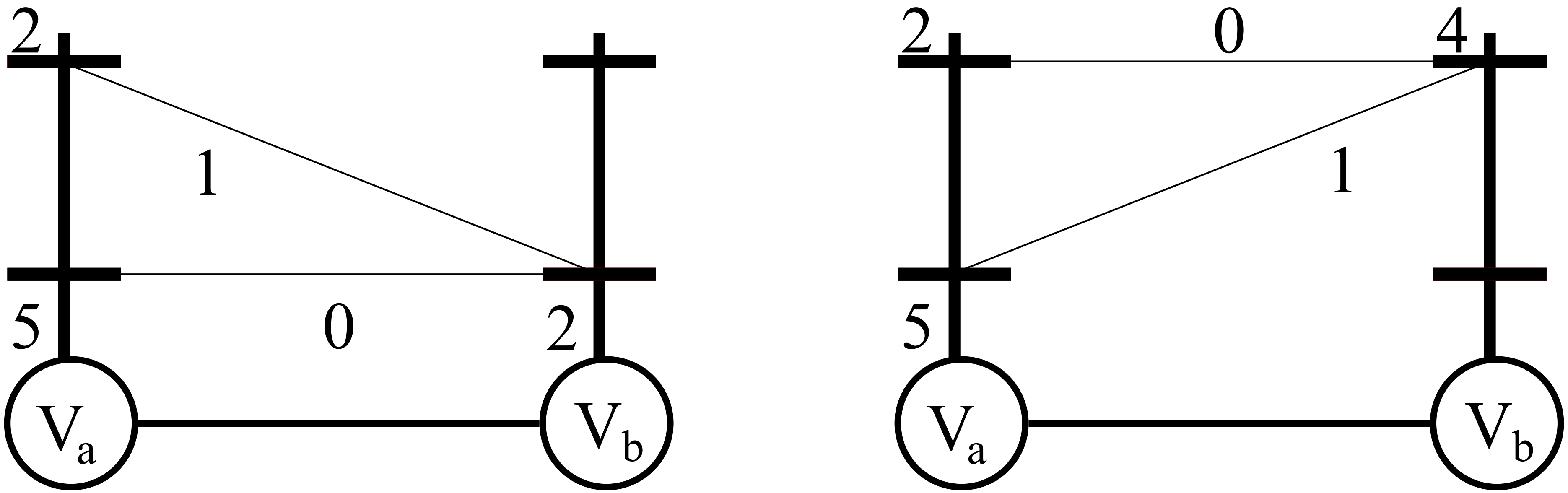
- Basics: problem formulation
  - Energy Function
  - MAP Estimation
  - Computing min-marginals
  - Reparameterization

- Solutions
  - Belief Propagation and related methods
  - Graph cuts

Belief Propagation

- Remember, some MAP problems are easy
- Belief Propagation gives exact MAP for chains
- Exact MAP for trees
- Clever Reparameterization

Two Variables

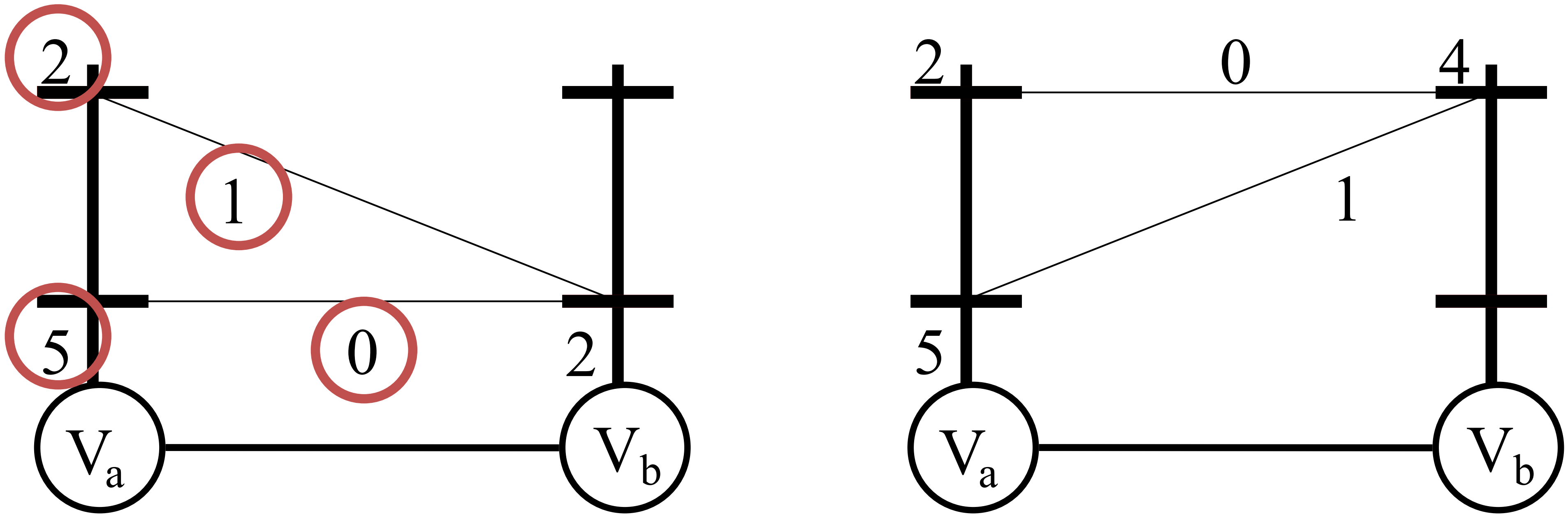


Add a constant to one $\theta_{b;k}$

Subtract that constant from $\theta_{ab;ik}$ for all 'i'

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

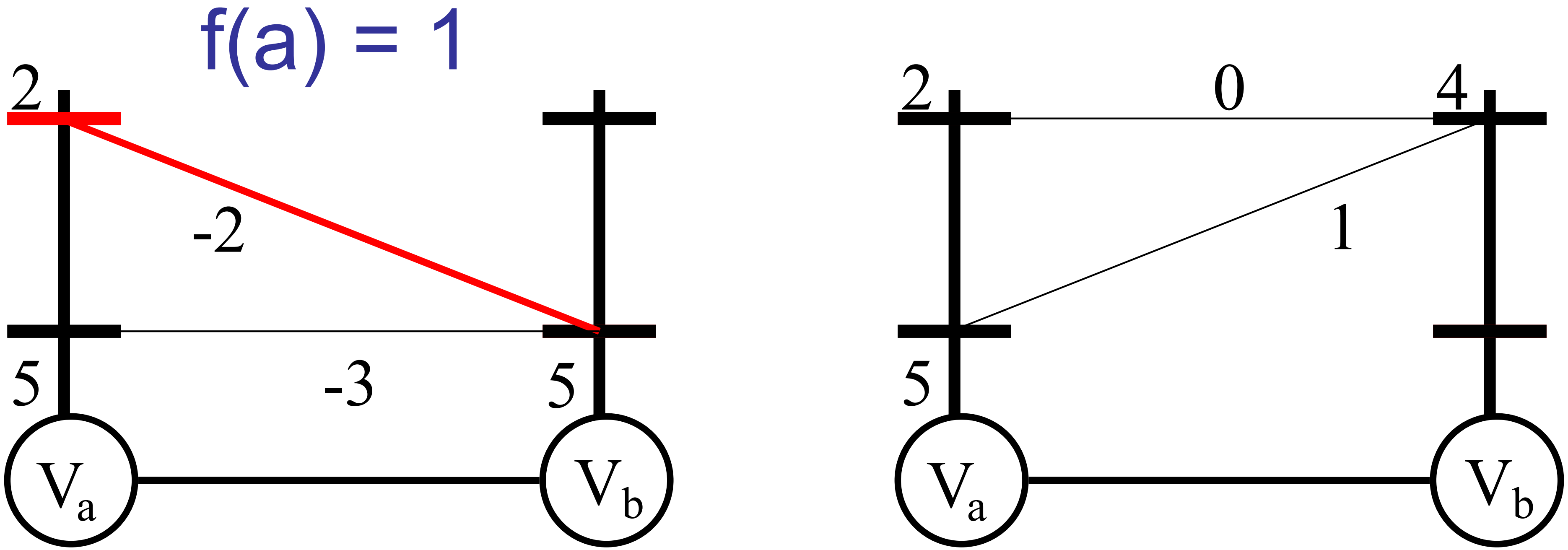
Two Variables



$$M_{ab;0} = \min \begin{matrix} \theta_{a;0} + \theta_{ab;00} = 5 + 0 \\ \theta_{a;1} + \theta_{ab;10} = 2 + 1 \end{matrix}$$

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

Two Variables

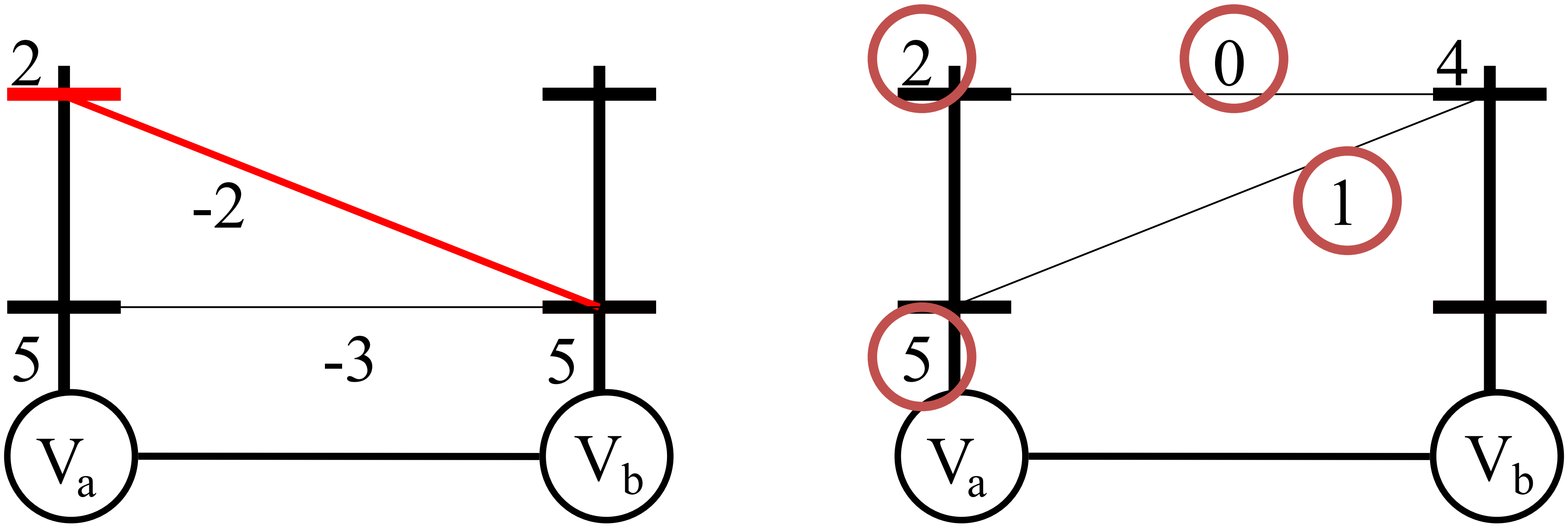


$$\theta'_{b;0} = q_{b;0}$$

Potentials along the red path add up to 0

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

Two Variables

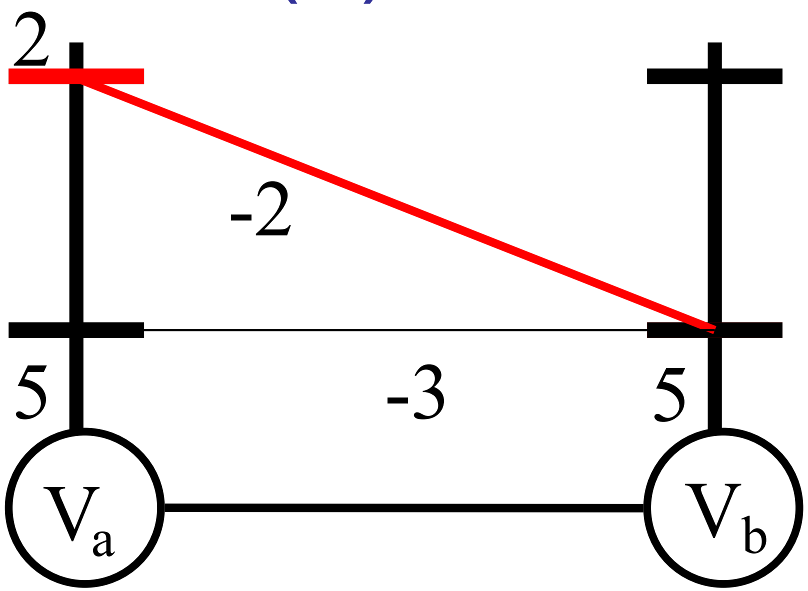


$$M_{ab;1} = \min \begin{array}{ll} \theta_{a;0} + \theta_{ab;01} & = 5 + 1 \\ \theta_{a;1} + \theta_{ab;11} & = 2 + 0 \end{array}$$

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

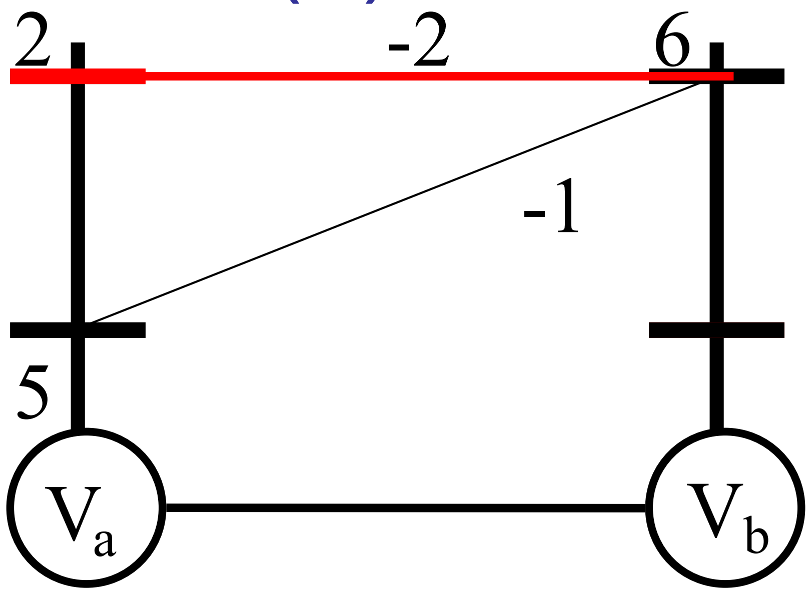
Two Variables

$f(a) = 1$



$\theta'_{b;0} = q_{b;0}$

$f(a) = 1$

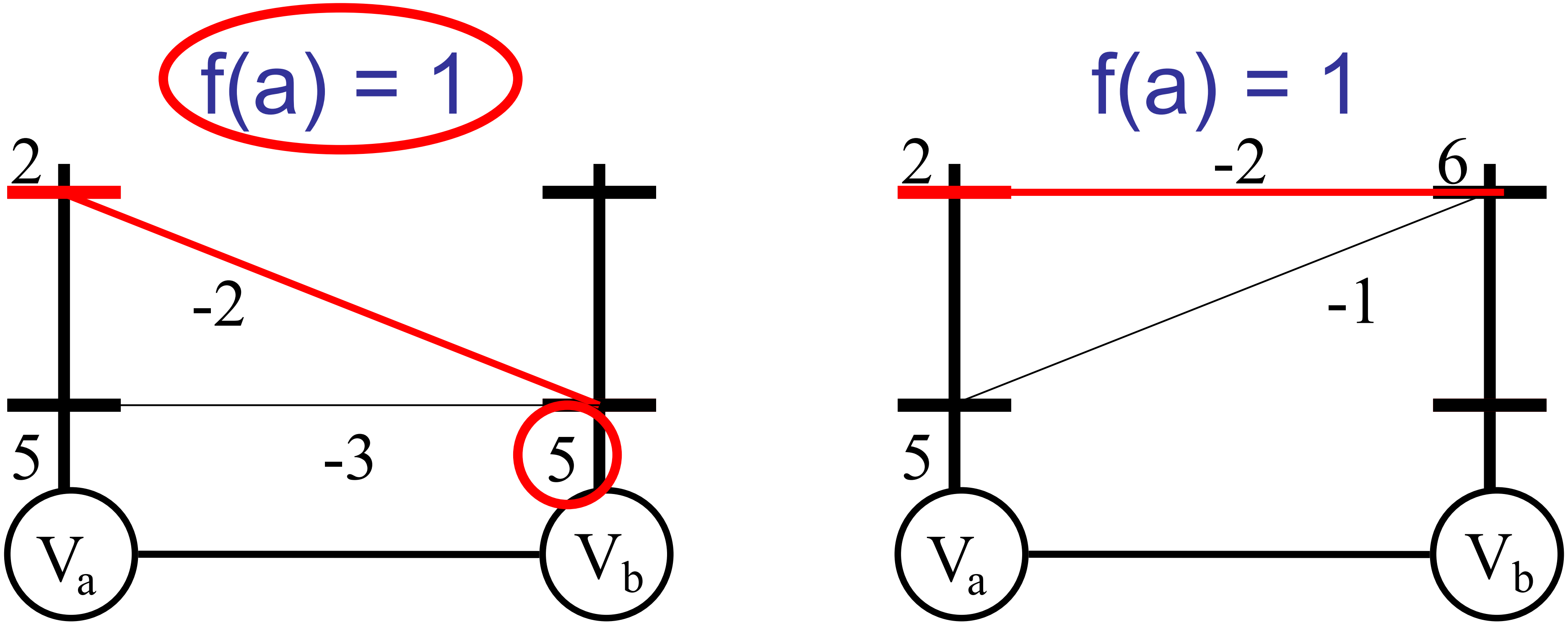


$\theta'_{b;1} = q_{b;1}$

Minimum of min-marginals = MAP estimate

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

Two Variables



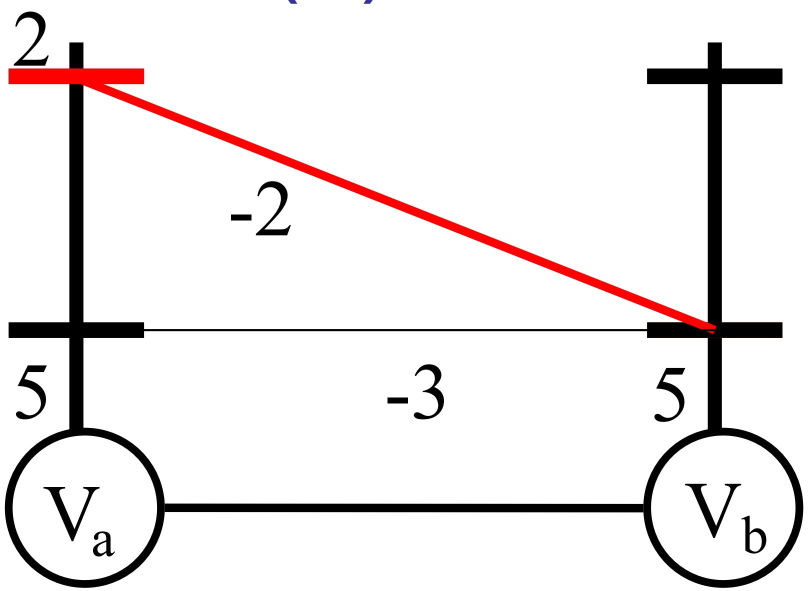
$\theta'_{b;0} = q_{b;0}$ $\qquad$ $\theta'_{b;1} = q_{b;1}$

f*(b) = 0 f*(a) = 1

Choose the ***right*** constant $\qquad \theta'_{b;k} = q_{b;k}$

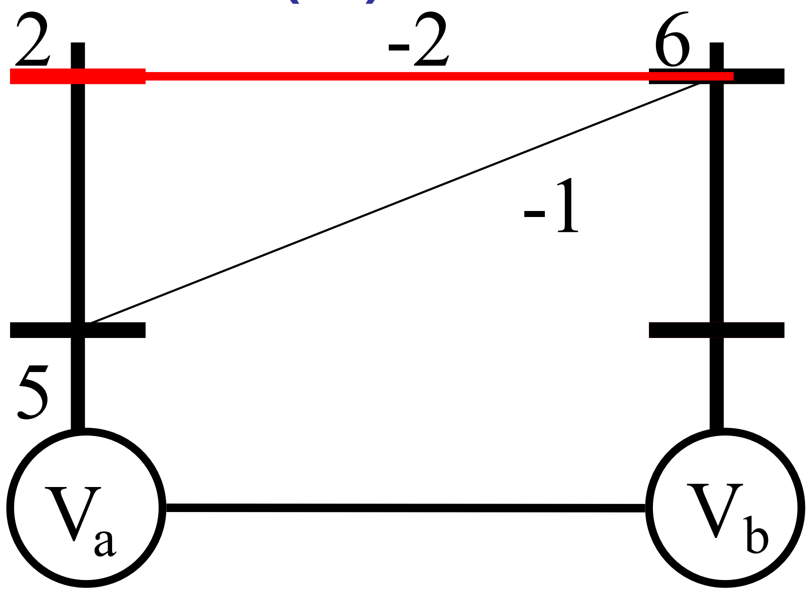
Two Variables

f(a) = 1



$\theta'_{b;0} = q_{b;0}$

f(a) = 1



$\theta'_{b;1} = q_{b;1}$

We get all the min-marginals of V_b

Choose the ***right*** constant $\theta'_{b;k} = q_{b;k}$

Recap

We only need to know two sets of equations

General form of Reparameterization

$$\theta'_{a;i} = \theta_{a;i} + M_{ba;i} \qquad \theta'_{b;k} = \theta_{b;k} + M_{ab;k}$$

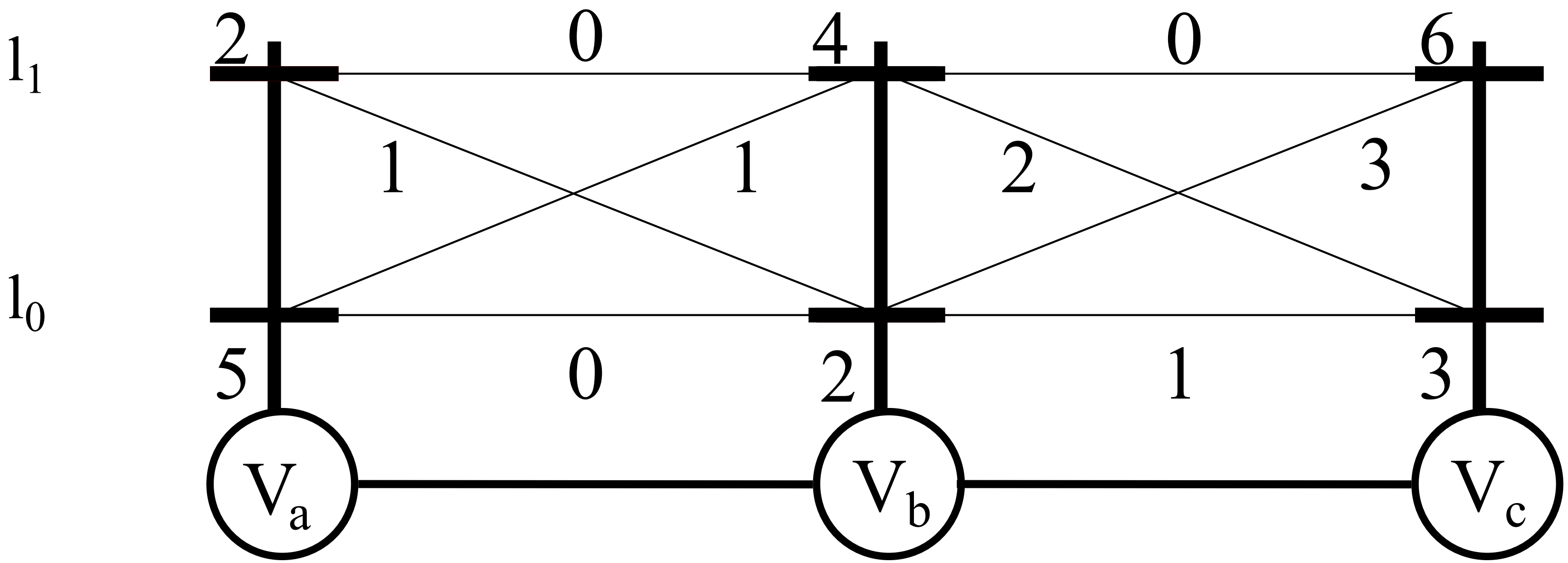
$$\theta'_{ab;ik} = \theta_{ab;ik} - M_{ab;k} - M_{ba;i}$$

Reparameterization of (a,b) in Belief Propagation

$$M_{ab;k} = \min_i \{ \theta_{a;i} + \theta_{ab;ik} \}$$

$$M_{ba;i} = 0$$

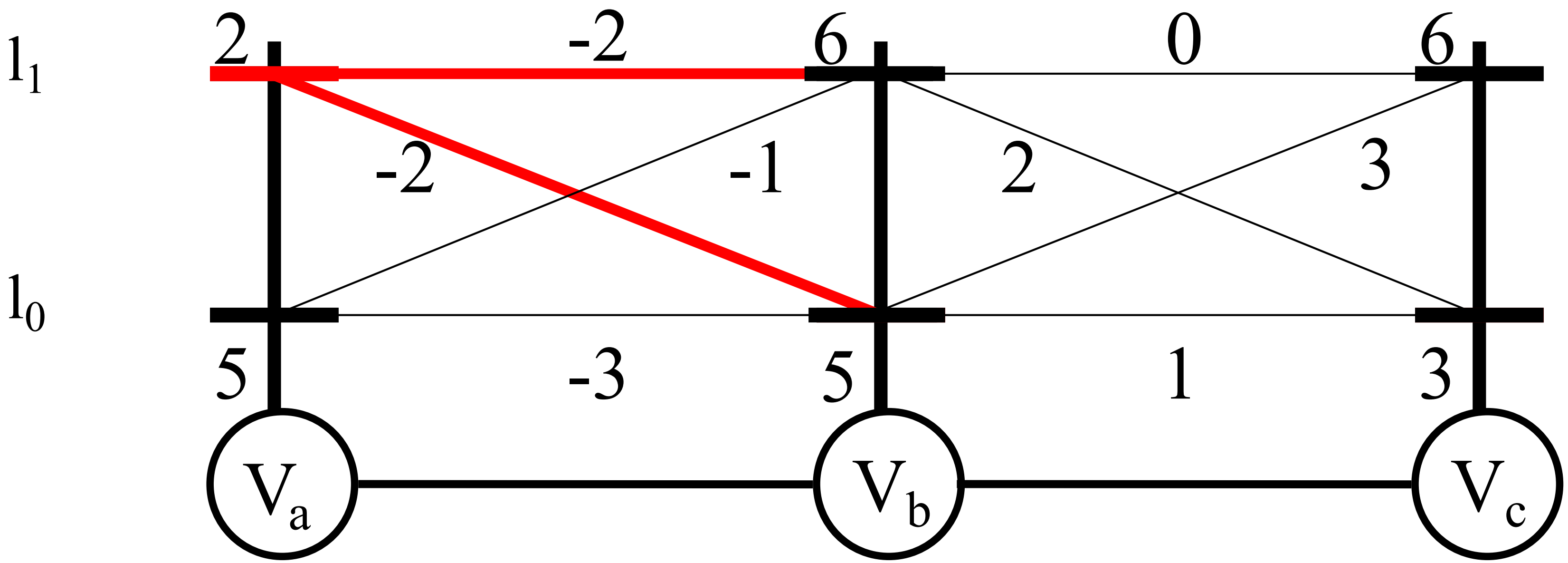
Three Variables



Reparameterize the edge (a,b) as before

Three Variables

f(a) = 1



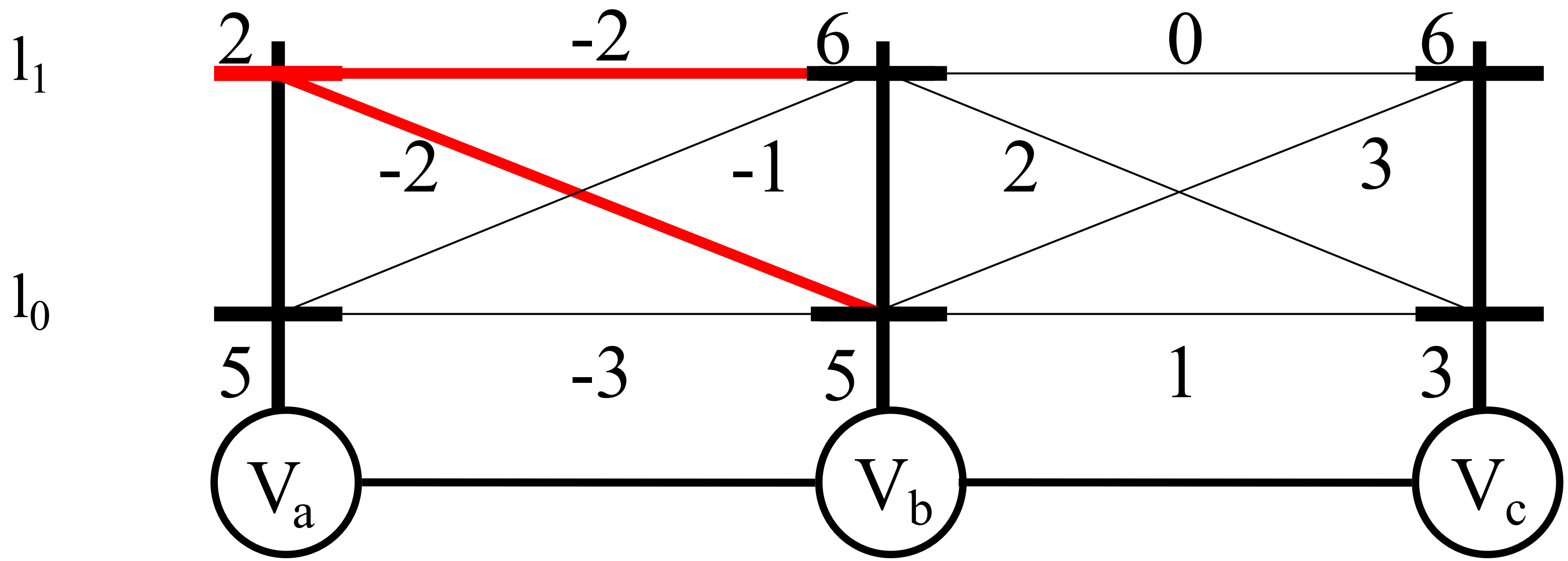
f(a) = 1

Reparameterize the edge (a,b) as before

Potentials along the red path add up to 0

Three Variables

f(a) = 1

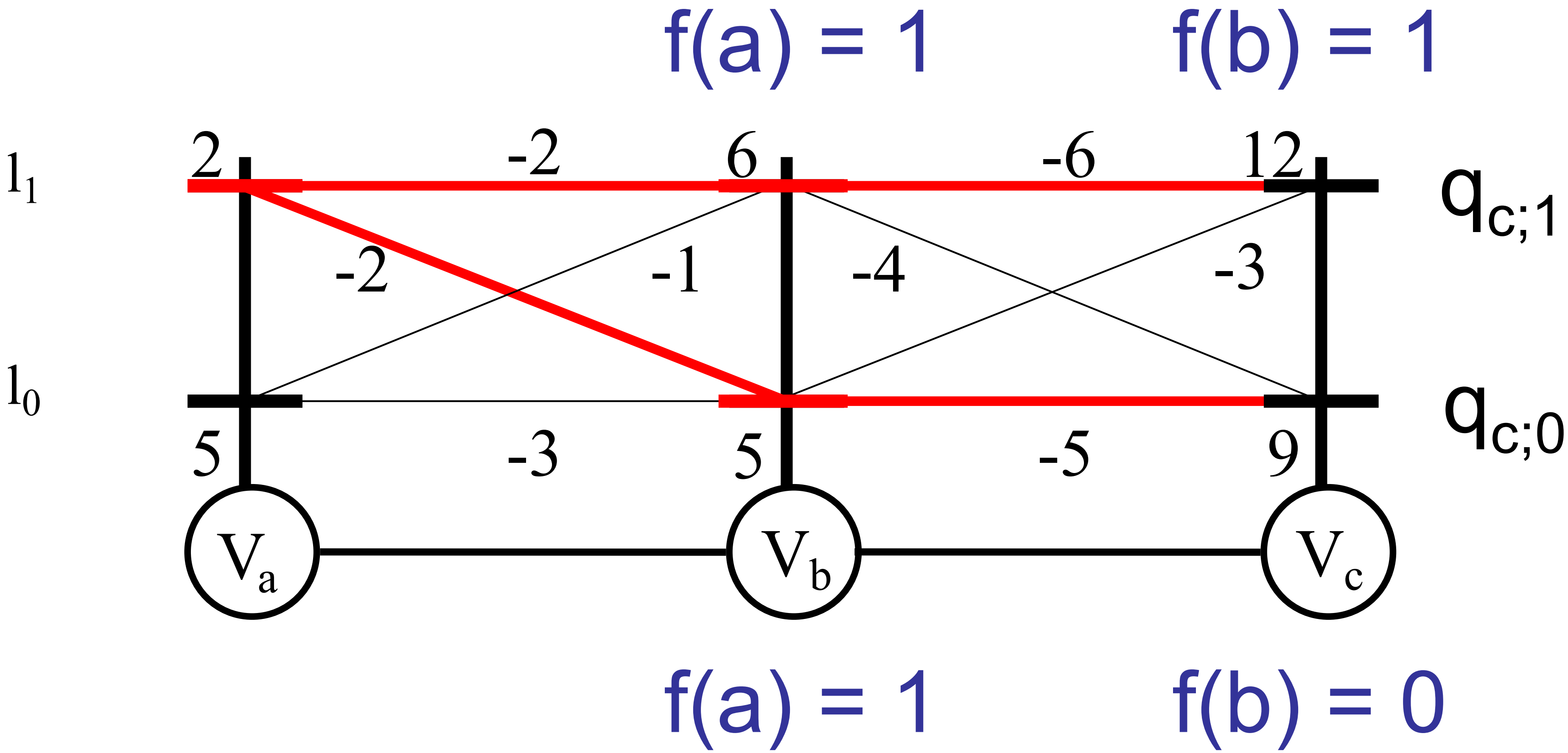


f(a) = 1

Reparameterize the edge (b,c) as before

Potentials along the red path add up to 0

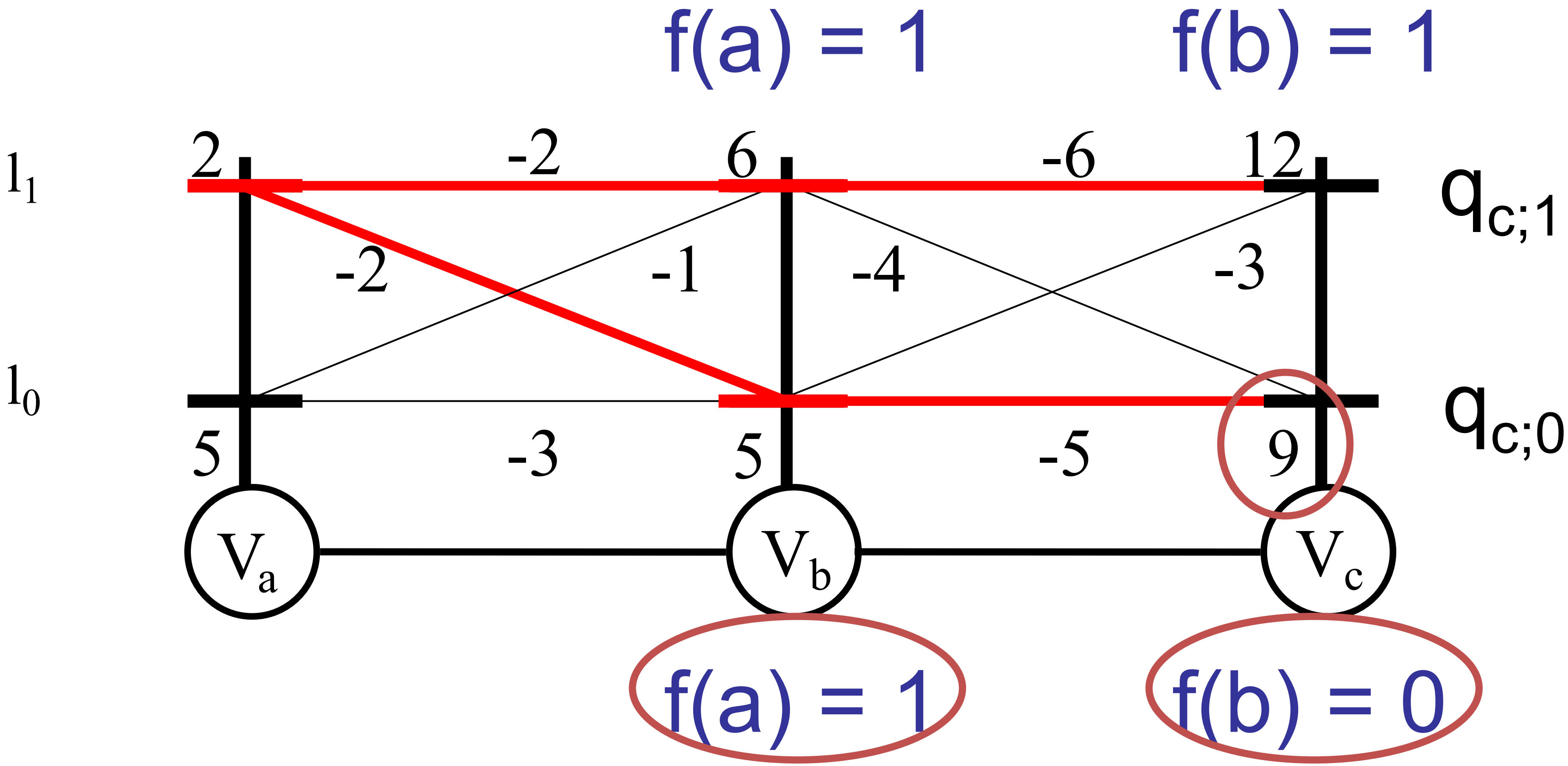
Three Variables



Reparameterize the edge (b,c) as before

Potentials along the red path add up to 0

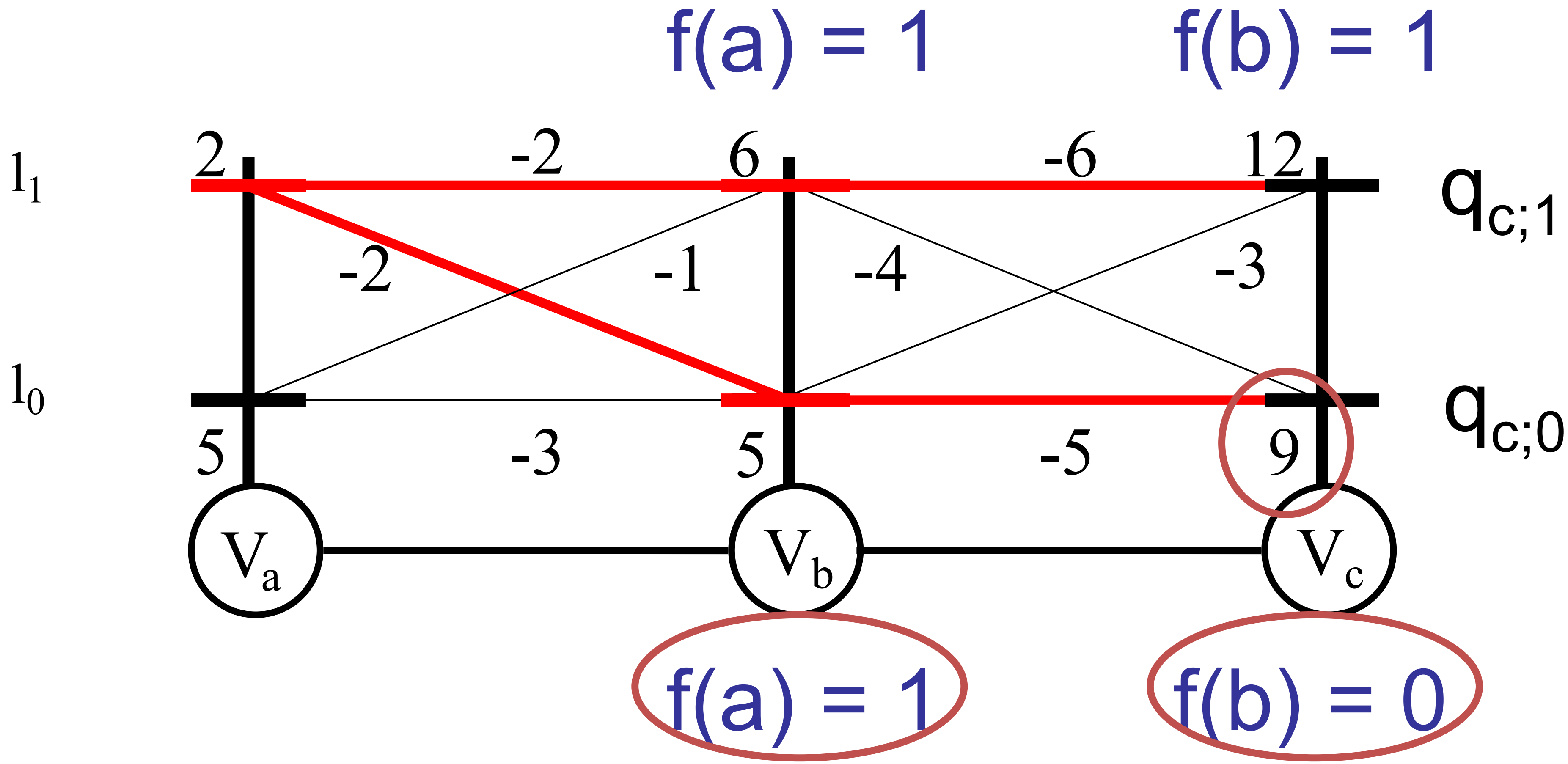
Three Variables



f*(c) = 0 f*(b) = 0 f*(a) = 1

Generalizes to any length chain

Three Variables



$f^*(c) = 0 \quad f^*(b) = 0 \quad f^*(a) = 1$

Only Dynamic Programming

Why Dynamic Programming?

3 variables $\equiv$ 2 variables + book-keeping

n variables $\equiv$ (n-1) variables + book-keeping

Start from left, go to right

Reparameterize current edge (a,b)

$$M_{ab;k} = \min_i \{ \theta_{a;i} + \theta_{ab;ik} \}$$

$$\theta'_{b;k} = \theta_{b;k} + M_{ab;k} \quad \theta'_{ab;ik} = \theta_{ab;ik} - M_{ab;k}$$

Repeat

Why Dynamic Programming?

Messages   Message Passing

Why stop at dynamic programming?

Start from left, go to right

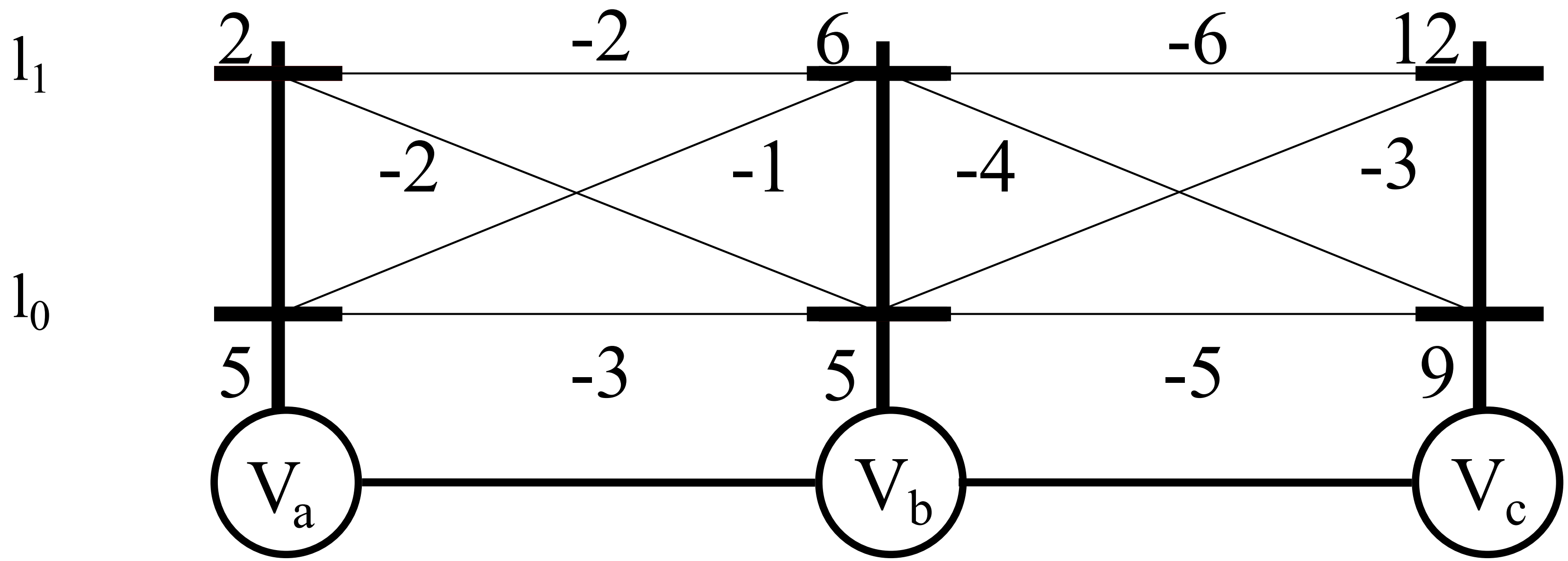
Reparameterize current edge (a,b)

$$M_{ab;k} = \min_i \{ \theta_{a;i} + \theta_{ab;ik} \}$$

$$\theta'_{b;k} = \theta_{b;k} + M_{ab;k} \quad \theta'_{ab;ik} = \theta_{ab;ik} - M_{ab;k}$$

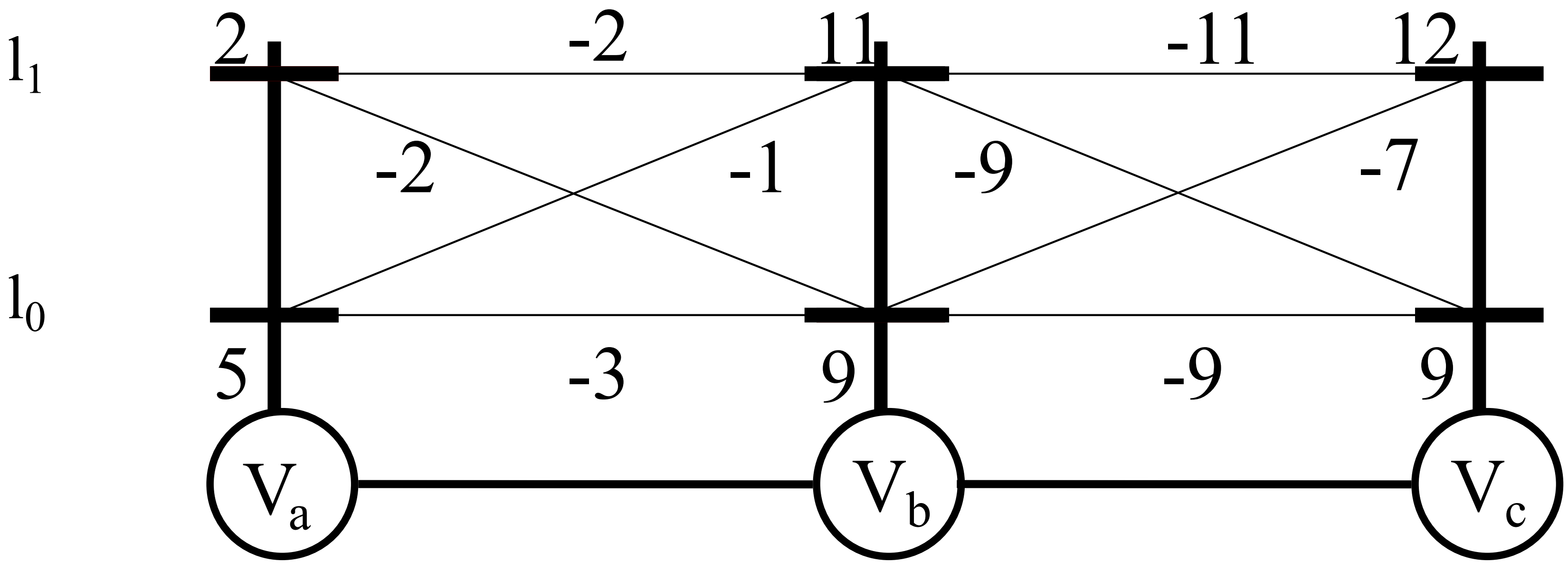
Repeat

Three Variables



Reparameterize the edge (c,b) as before

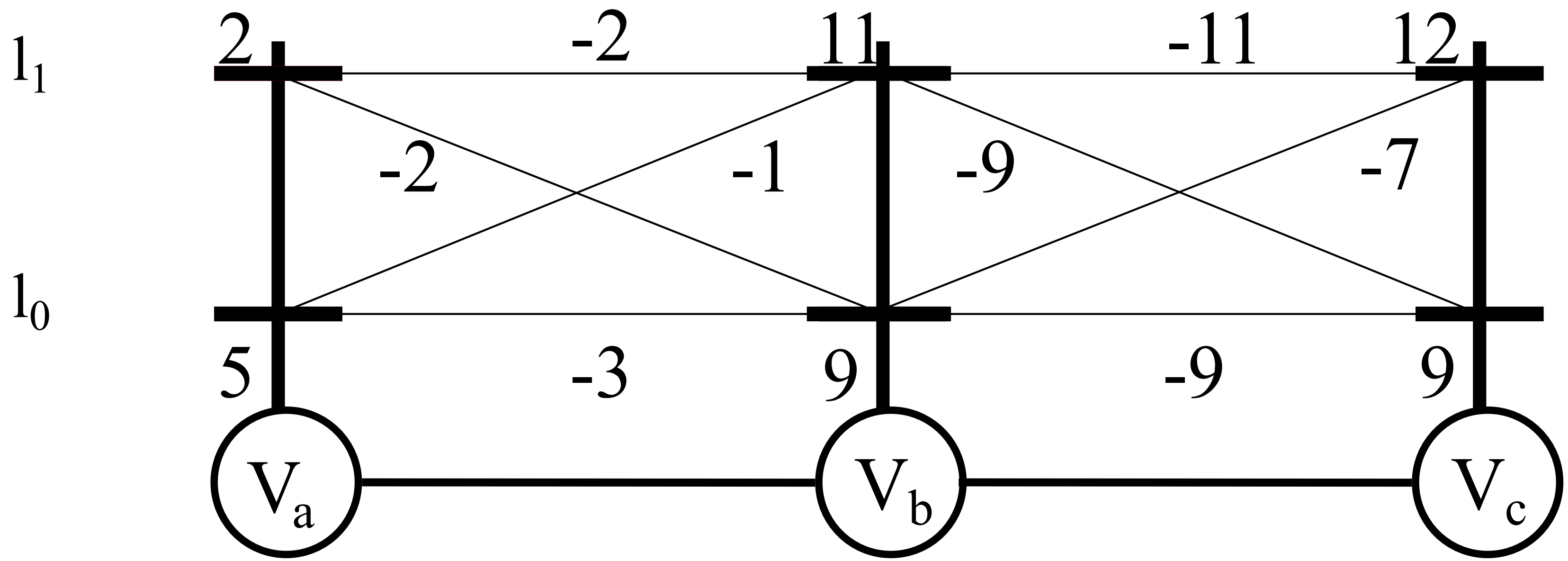
Three Variables



Reparameterize the edge (c,b) as before

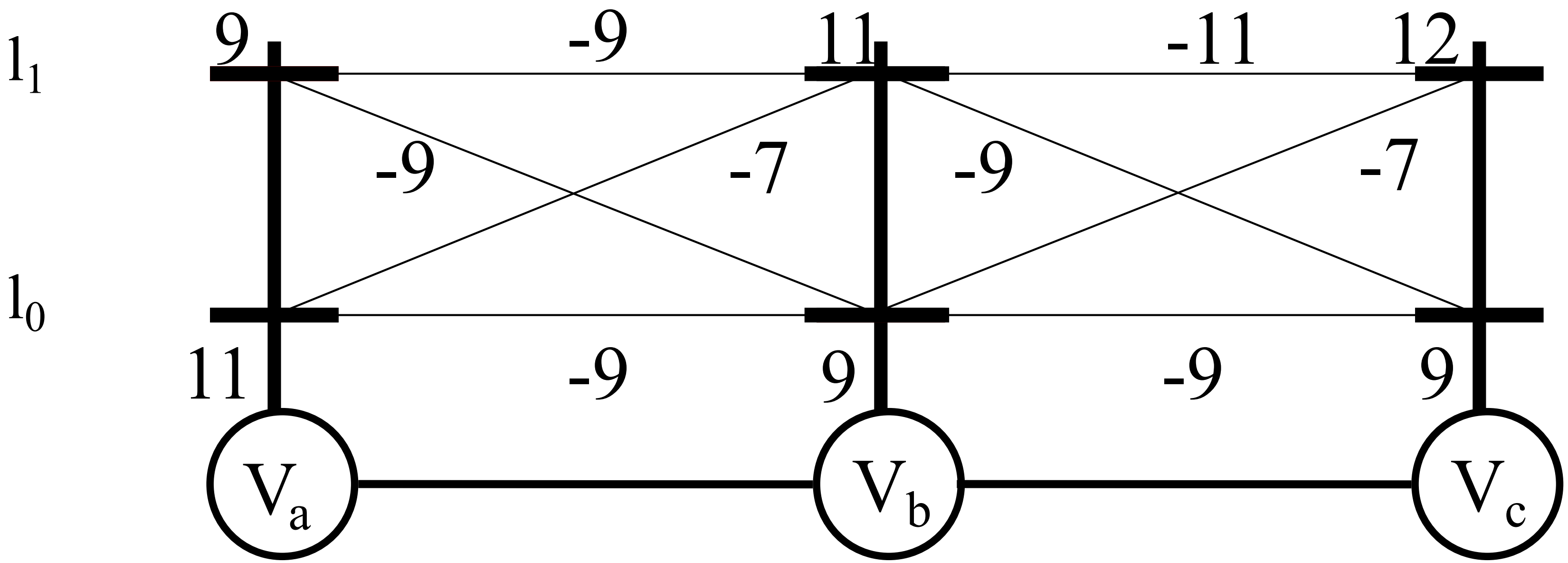
$$\theta'_{b;i} = q_{b;i}$$

Three Variables



Reparameterize the edge (b,a) as before

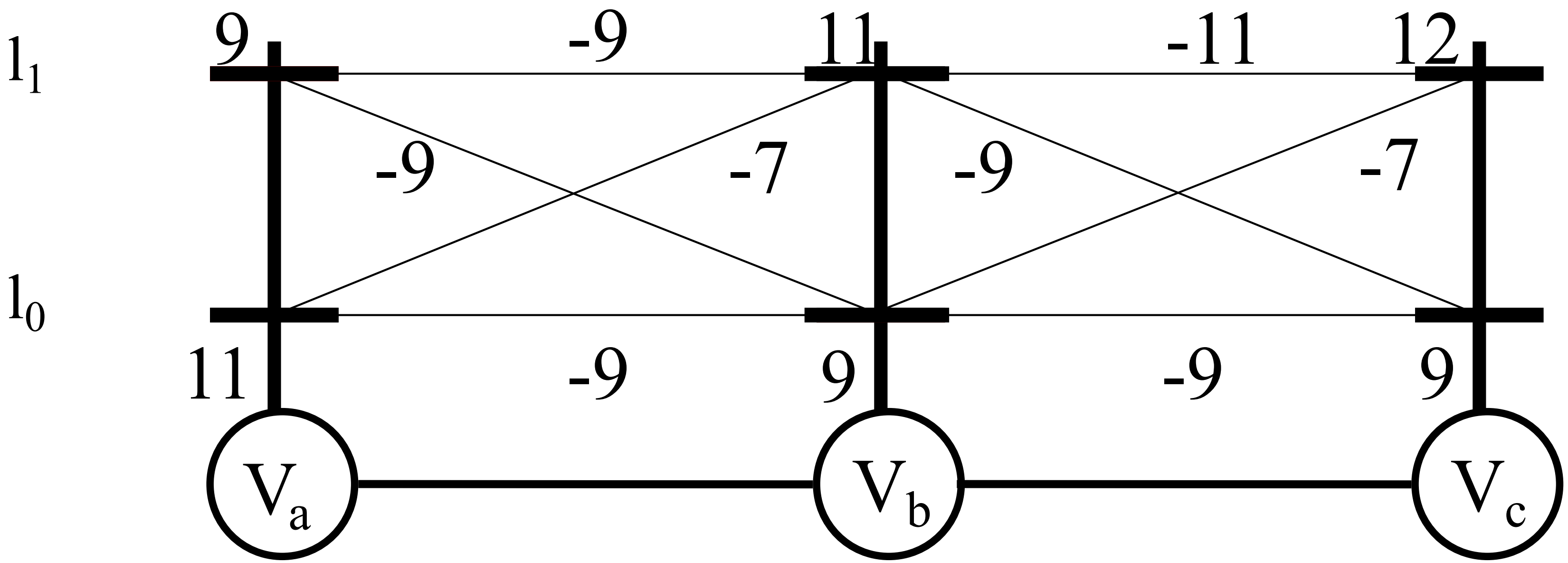
Three Variables



Reparameterize the edge (b,a) as before

$$\theta'_{a;i} = q_{a;i}$$

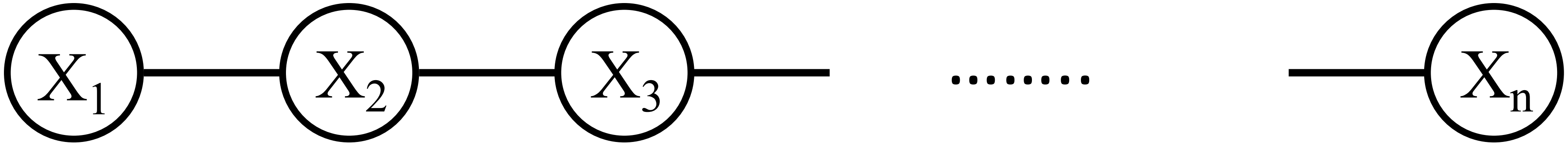
Three Variables



Forward Pass → ← Backward Pass

All min-marginals are computed

Chains



Reparameterize the edge (1,2)

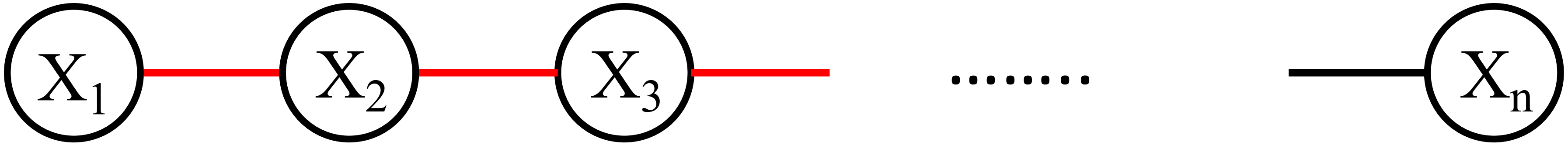
Chains

Reparameterize the edge (1,2)

Chains

Reparameterize the edge (2,3)

Chains



Reparameterize the edge (3,4)

Chains

Reparameterize the edge (n-1,n)

Min-marginals $e_n(i)$ for all labels

Belief Propagation on Chains

Start from left, go to right

Reparameterize current edge (a,b)

$$M_{ab;k} = \min_i \{ \theta_{a;i} + \theta_{ab;ik} \}$$

$$\theta'_{b;k} = \theta_{b;k} + M_{ab;k} \quad \theta'_{ab;ik} = \theta_{ab;ik} - M_{ab;k}$$

Repeat till the end of the chain

Start from right, go to left

Repeat till the end of the chain

Belief Propagation on Chains

- Generalizes to chains of any length
- A way of computing reparam constants
- Forward Pass - Start to End
  - MAP estimate
  - Min-marginals of final variable
- Backward Pass - End to start
  - All other min-marginals

Computational Complexity

Number of reparameterization constants = $(n-1)h$

Complexity for each constant = $O(h)$

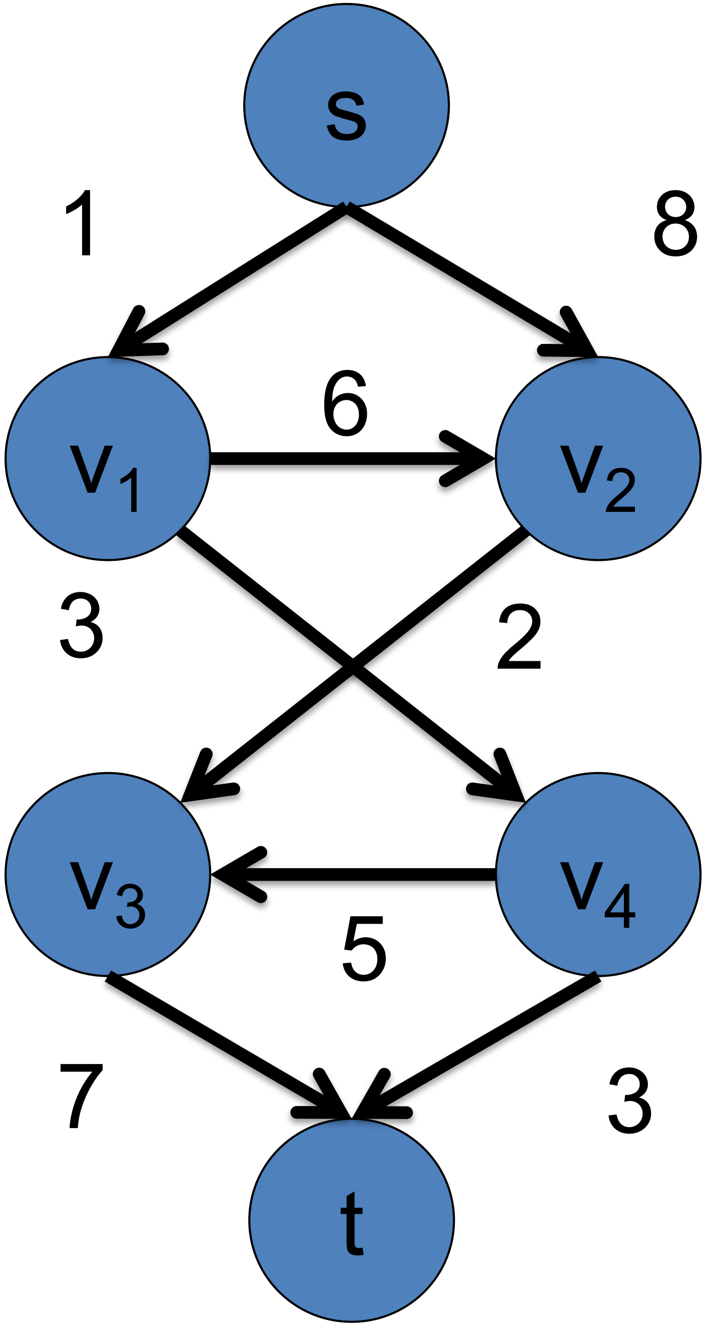
Total complexity = $O(nh^2)$

Better than brute-force $O(h^n)$

Outline

- Preliminaries
  - **s-t Flow**
  - s-t Cut
  - Flows vs. Cuts
- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

s-t Flow



Function flow: A ➔ R

Flow of arc ≤ arc capacity

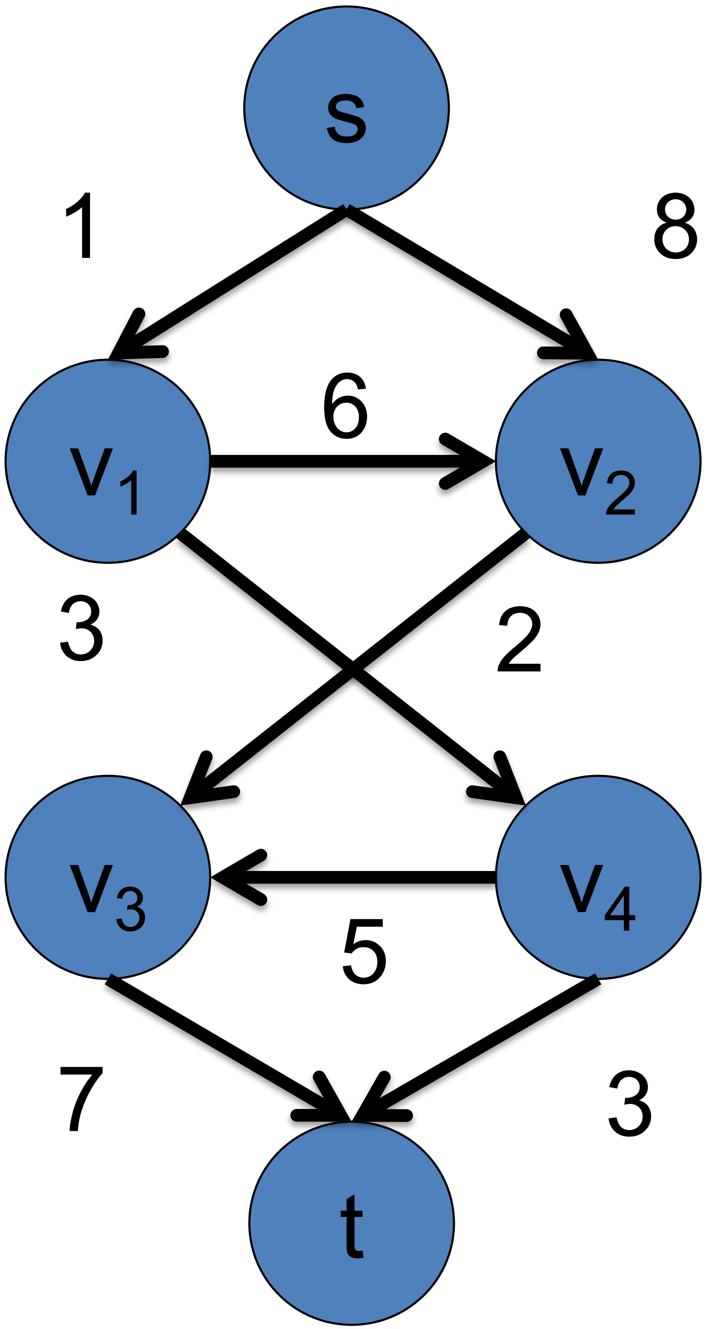
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A → R

flow(a) ≤ c(a)

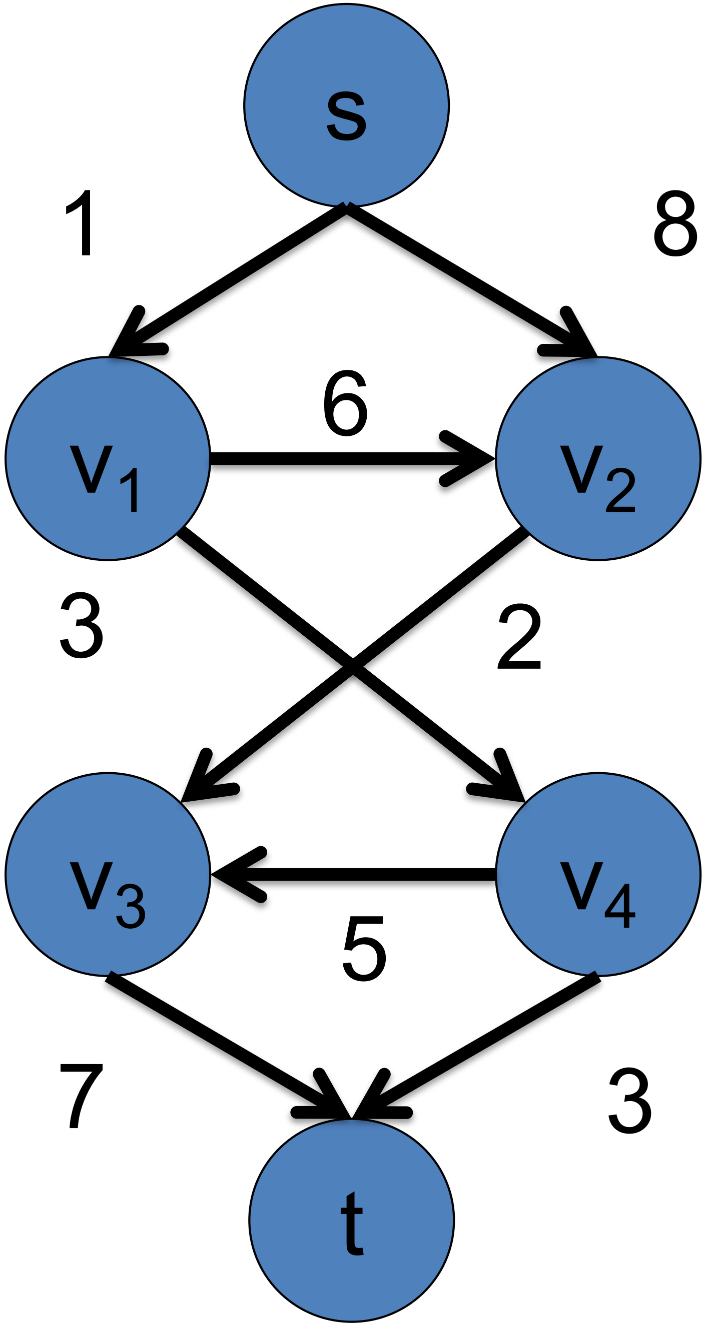
Flow is non-negative

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A → R

flow(a) ≤ c(a)

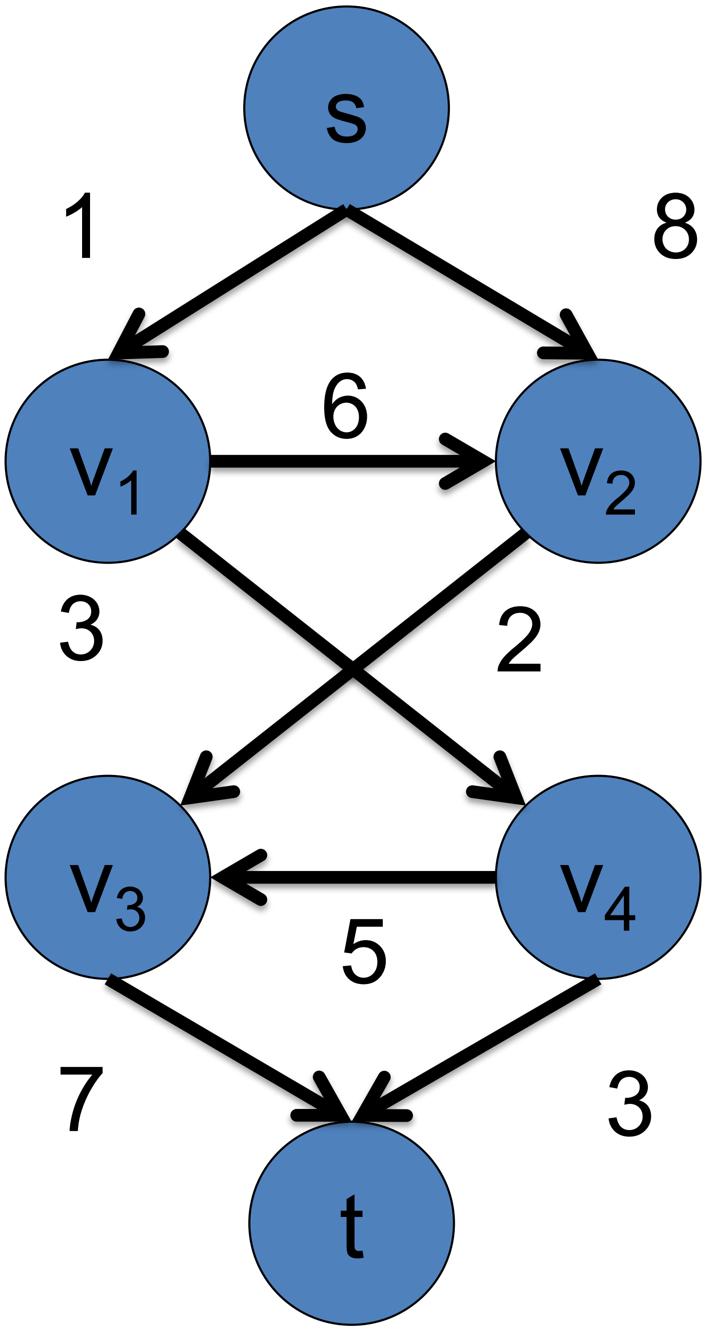
flow(a) ≥ 0

For all vertex except s,t

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

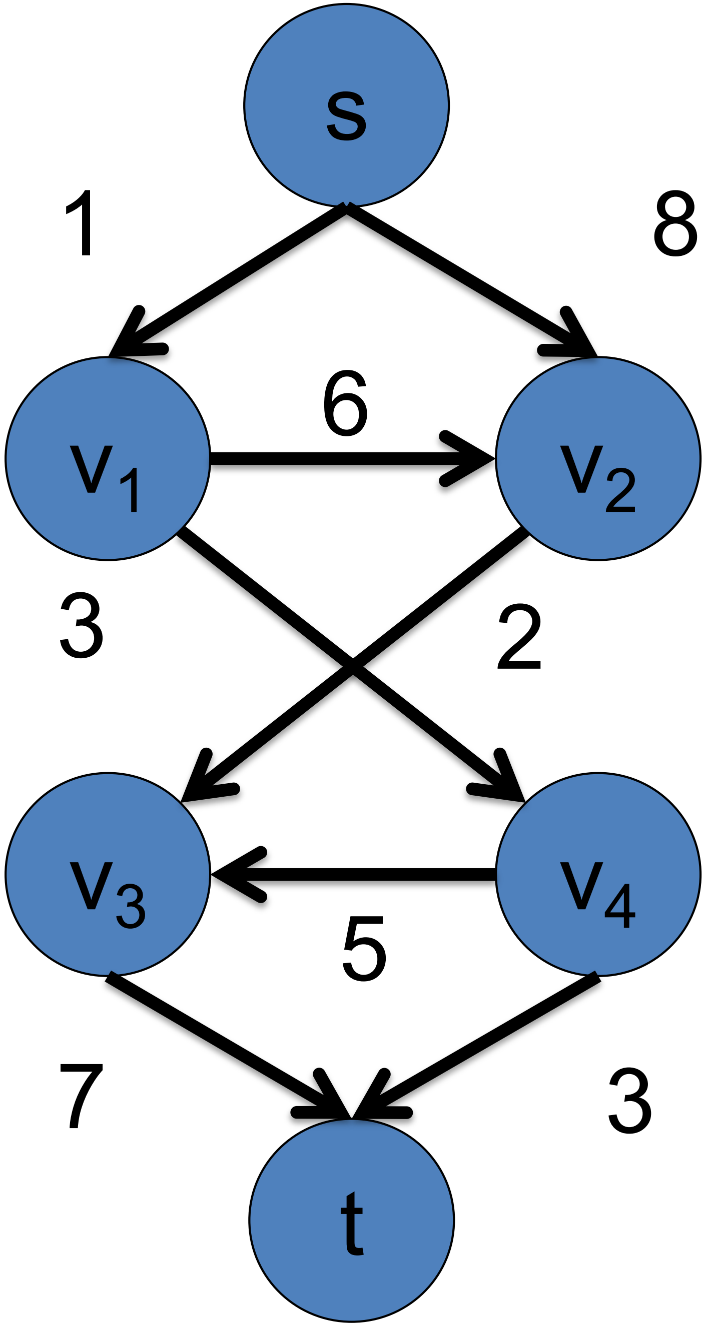
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

Incoming flow

= Outgoing flow

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

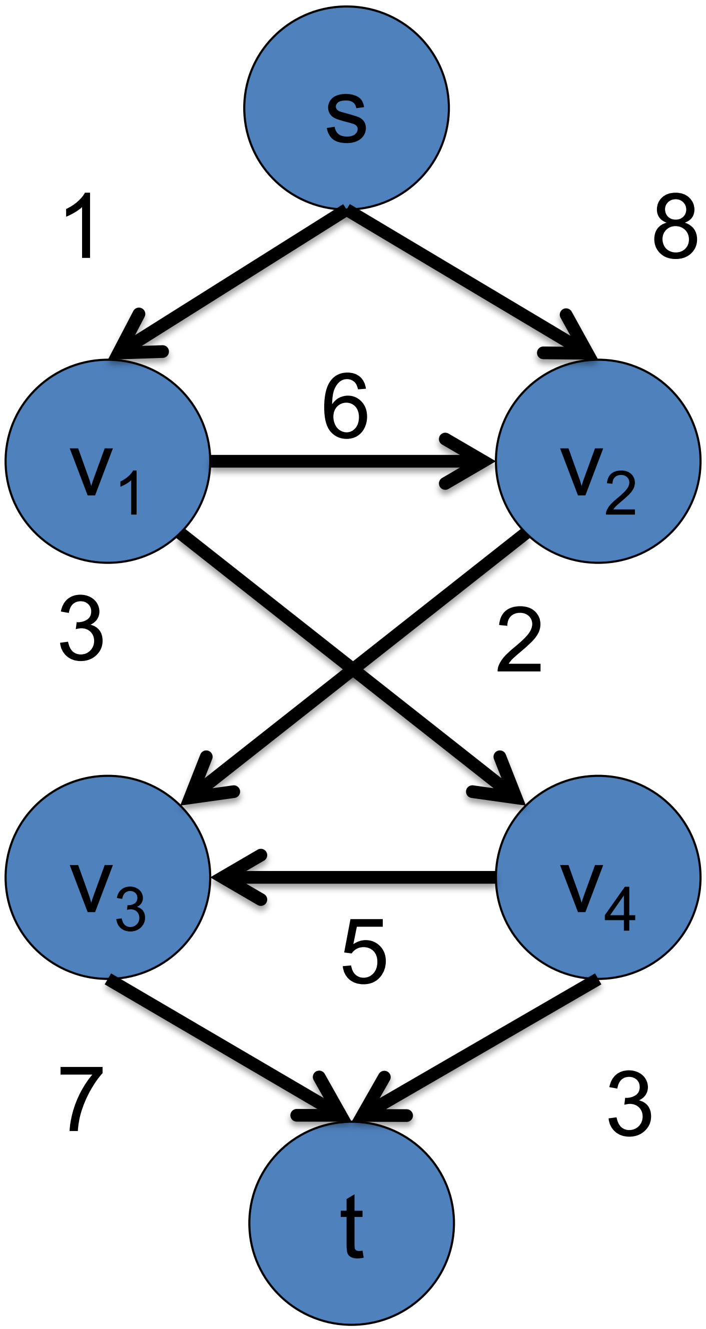
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$\Sigma_{(u,v)\in A}$ flow((u,v))

= Outgoing flow

s-t Flow



Function flow: A → R

$\text{flow}(a) \leq c(a)$

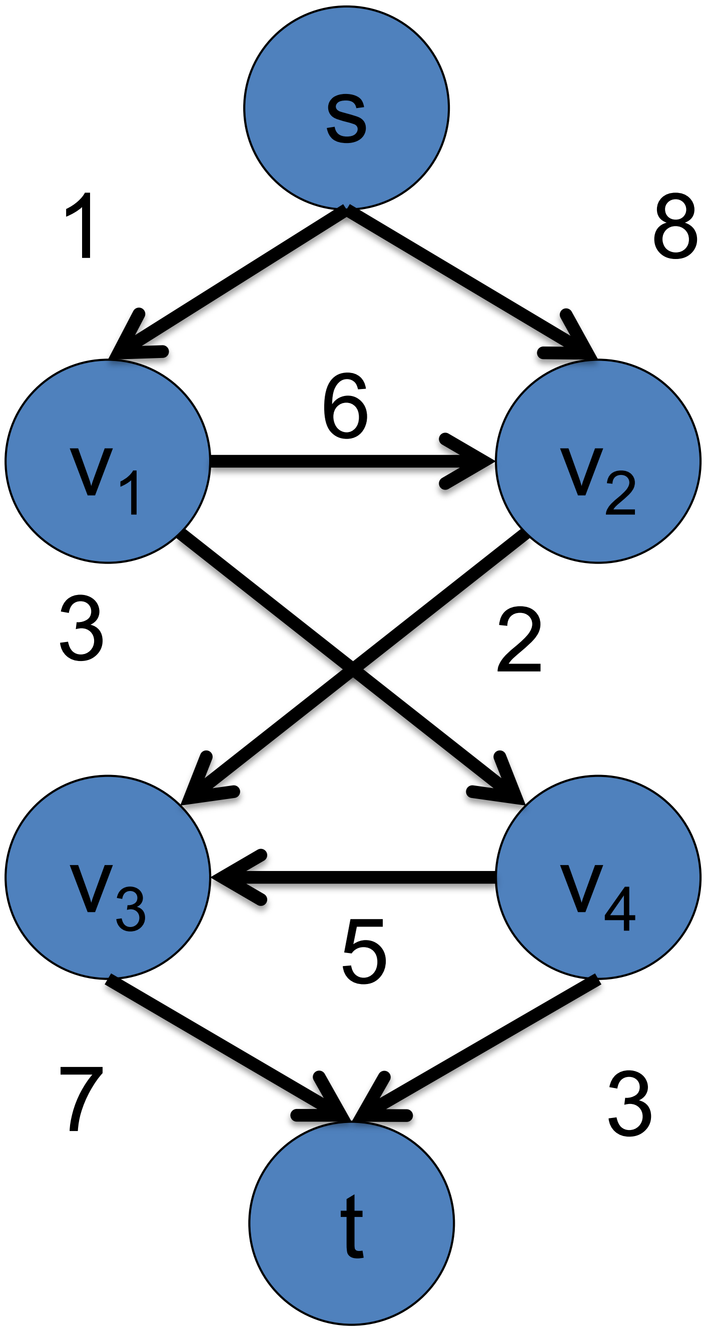
$\text{flow}(a) \geq 0$

For all $v \in V \setminus \{s,t\}$

$$\Sigma_{(u,v)\in A} \text{flow}((u,v))$$

$$= \Sigma_{(v,u)\in A} \text{flow}((v,u))$$

s-t Flow



Function flow: A ➔ R

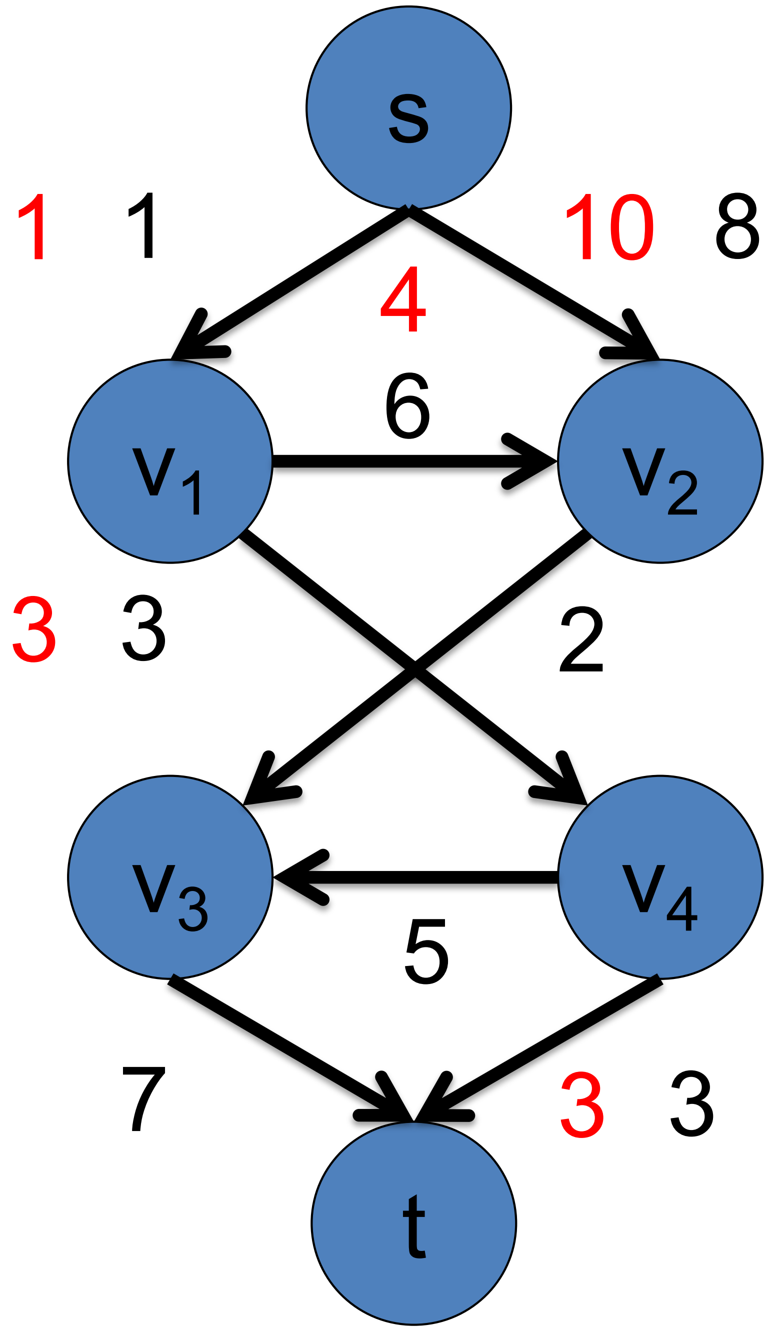
flow(a) ≤ c(a)

flow(a) ≥ 0

For all v ∈ V \ {s,t}

$E_{flow}(v) = 0$

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

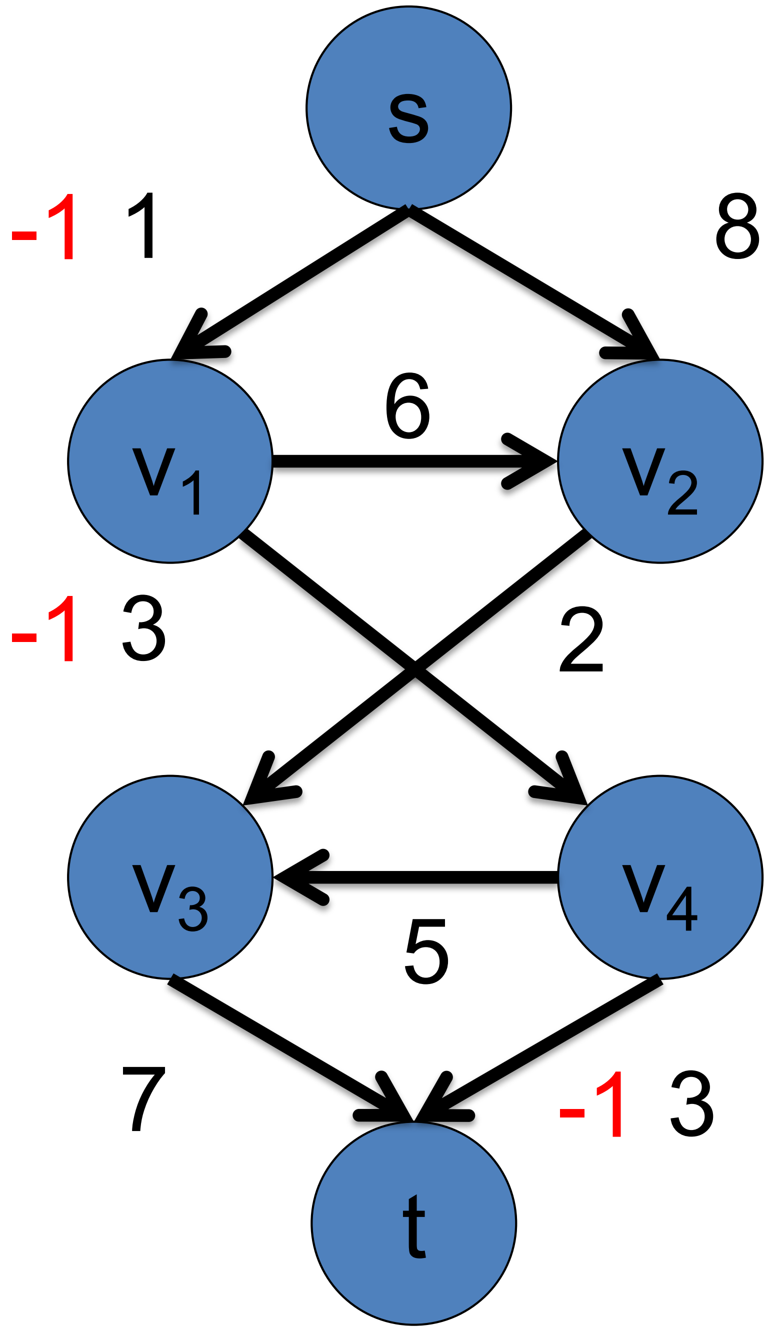
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

X

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

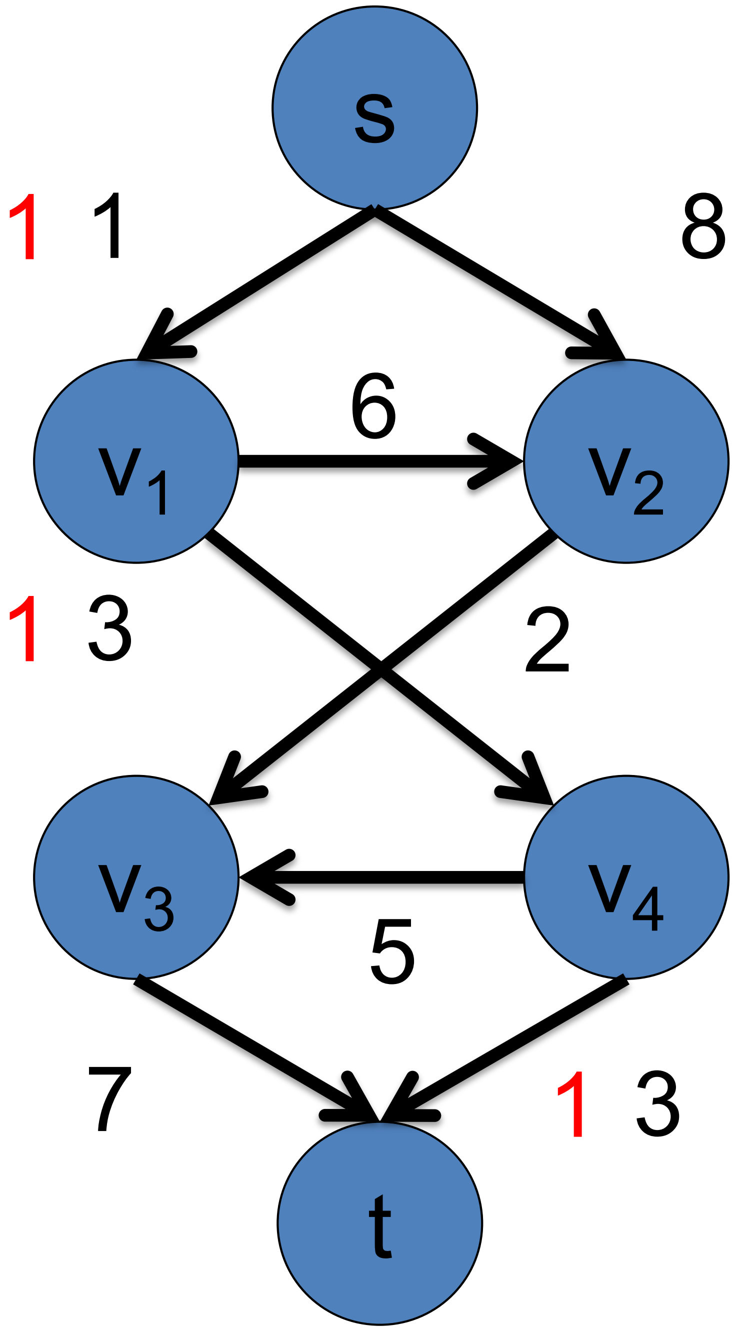
flow(a) ≥ 0

For all v ∈ V \ {s,t}

$E_{flow}(v) = 0$

X

s-t Flow



Function flow: A ➔ R

flow(a) ≤ c(a)

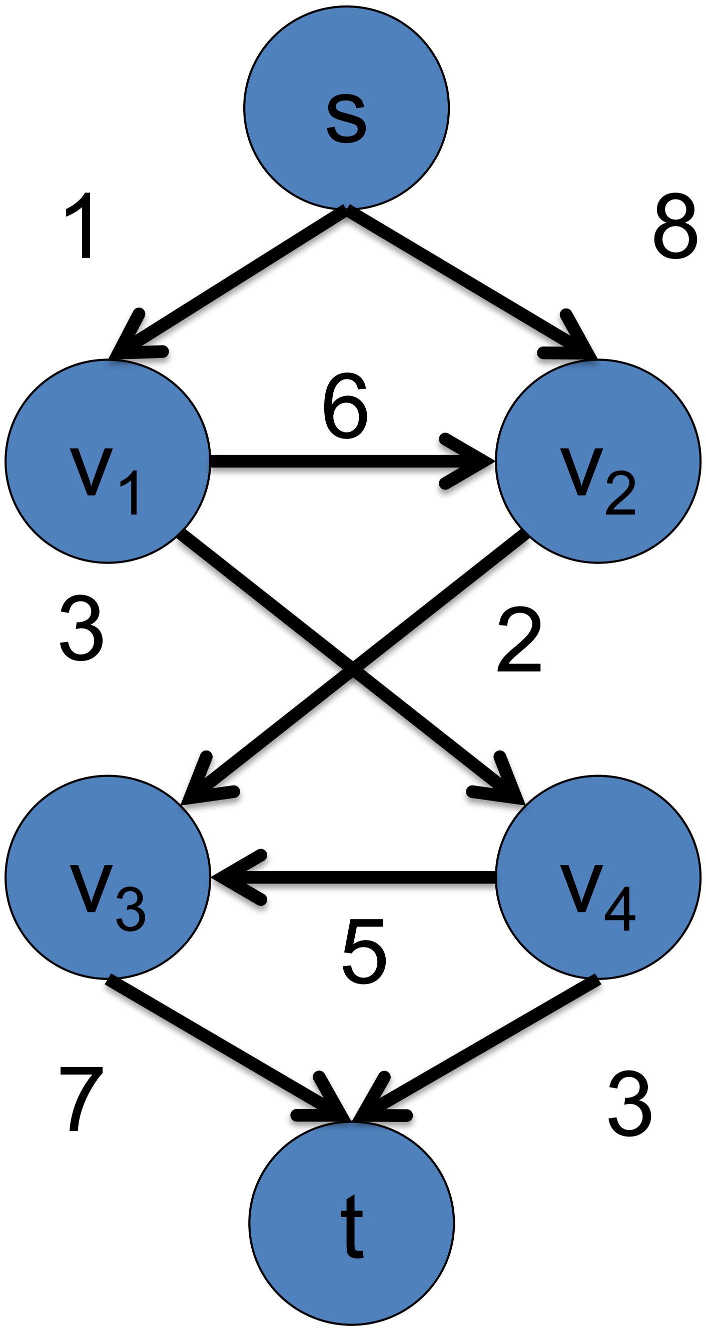
flow(a) ≥ 0

For all $v \in V \setminus \{s,t\}$

$E_{flow}(v) = 0$

✓

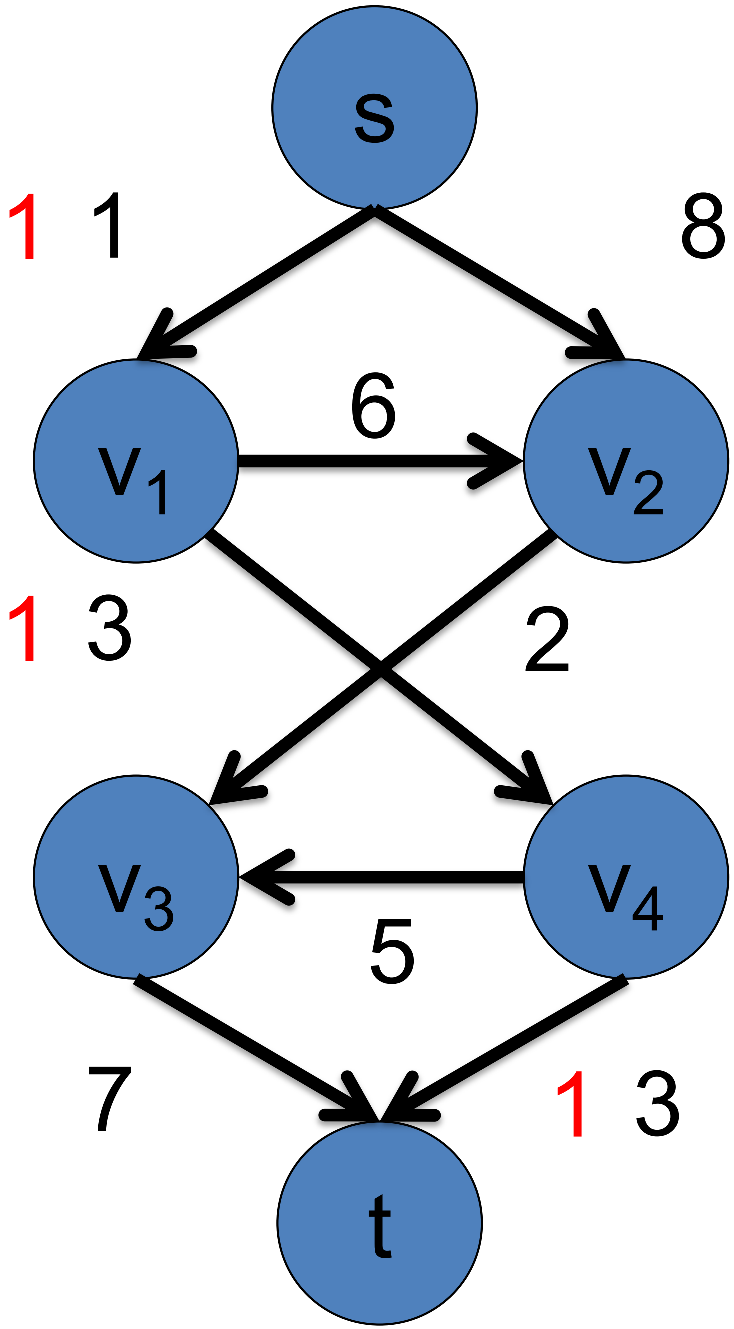
Value of s-t Flow



Outgoing flow of s

- Incoming flow of s

Value of s-t Flow



$-E_{flow}(s)$ $\quad E_{flow}(t)$

$$\Sigma_{(s,v)\in A}\ \mathrm{flow}((s,v))$$

$$-\ \Sigma_{(u,s)\in A}\ \mathrm{flow}((u,s))$$

Value = 1

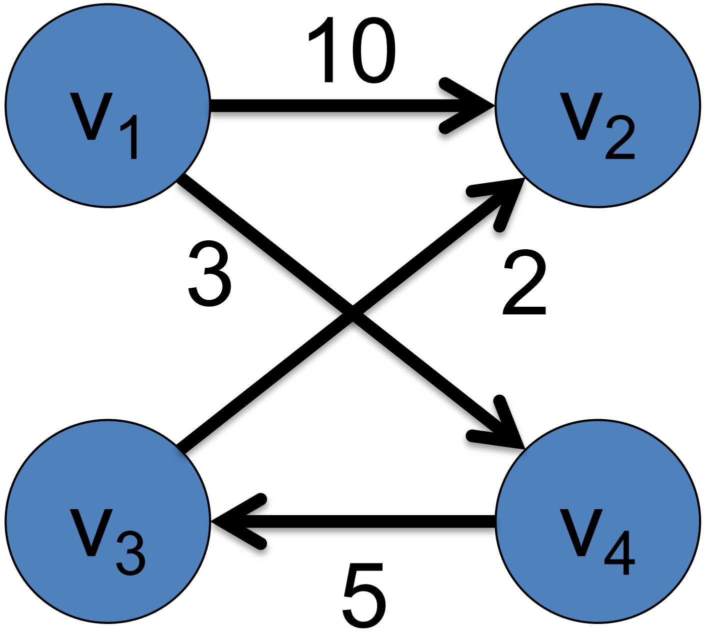
Outline

- Preliminaries
  - Functions and Excess Functions
  - s-t Flow
  - **s-t Cut**
  - Flows vs. Cuts
- Maximum Flow
- Algorithms
- Energy minimization with max flow/min cut

Cut

D = (V, A)

Let U be a subset of V

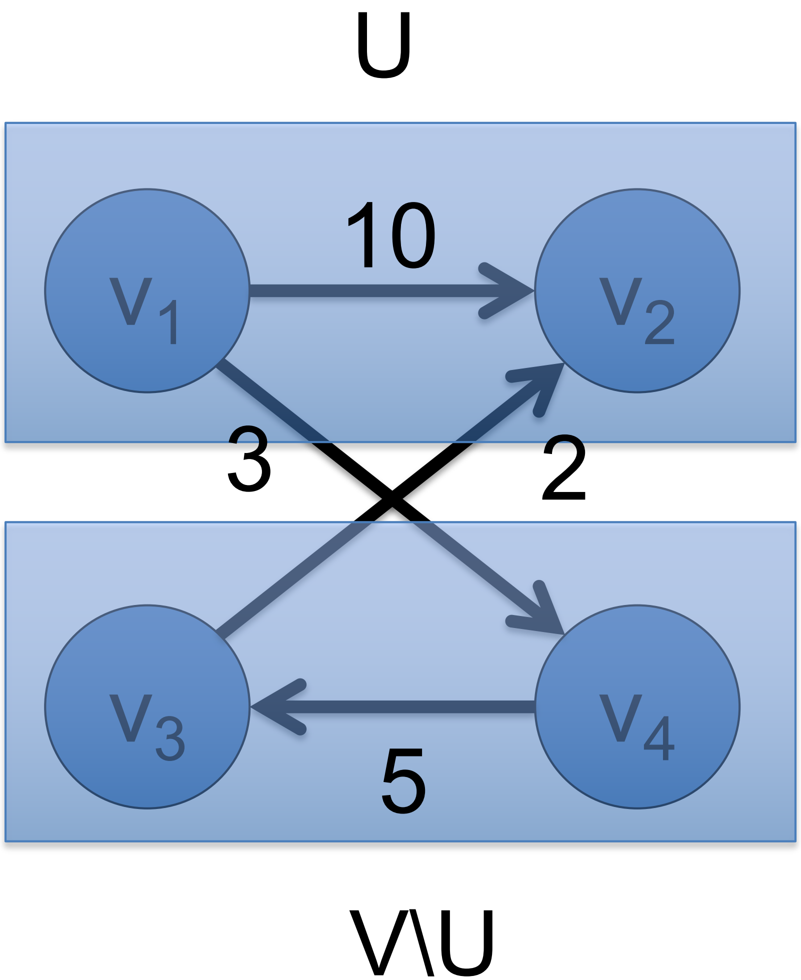


C is a set of arcs such that

- $(u,v) \in A$
- $u \in U$
- $v \in V \backslash U$

C is a cut in the digraph D

Cut



D = (V, A)

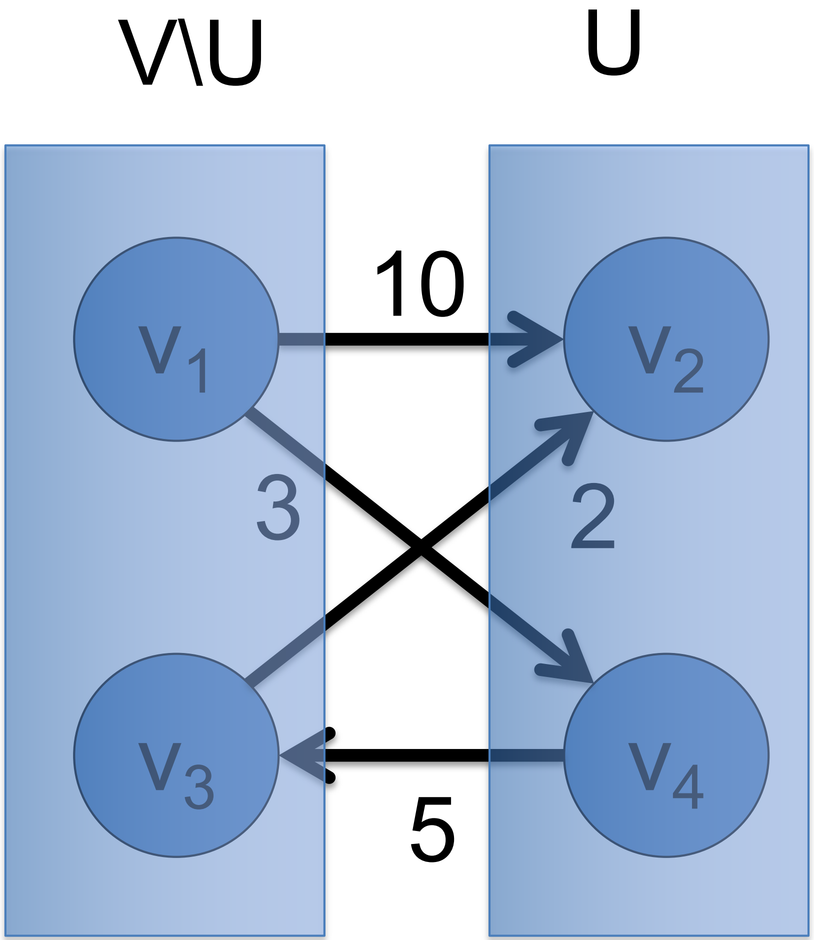
What is C?

$\{(v_1,v_2),(v_1,v_4)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

✓ $\{(v_1,v_4)\}$?

Cut



D = (V, A)

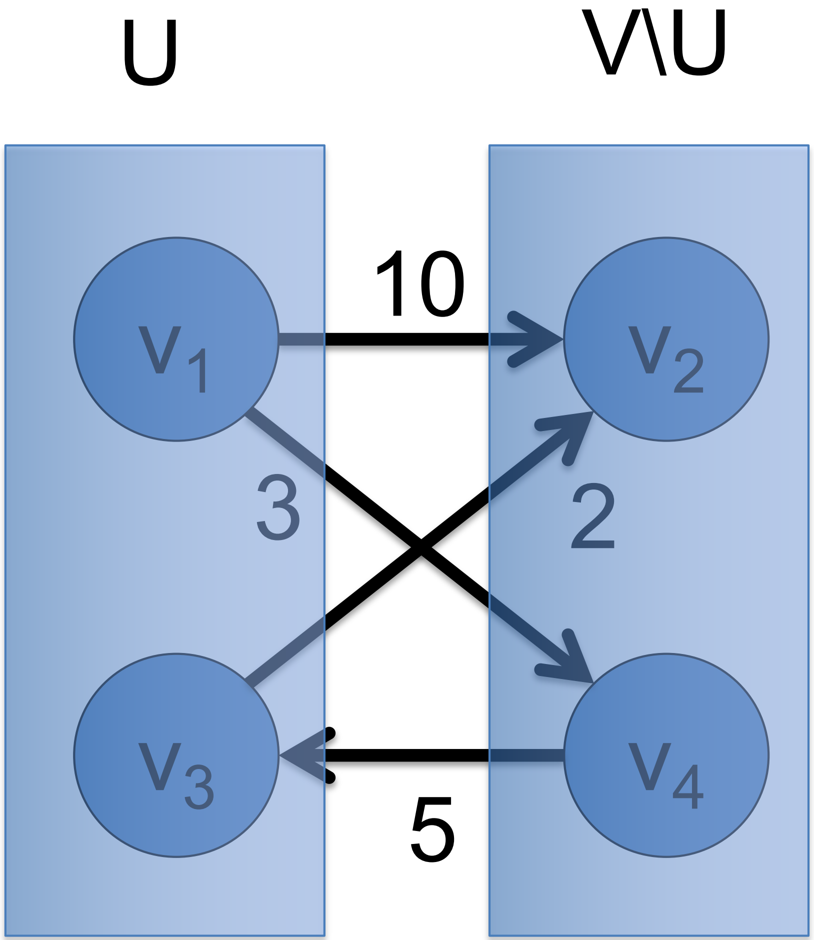
What is C?

$\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$?

✓ $\{(v_4,v_3)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

Cut



D = (V, A)

What is C?

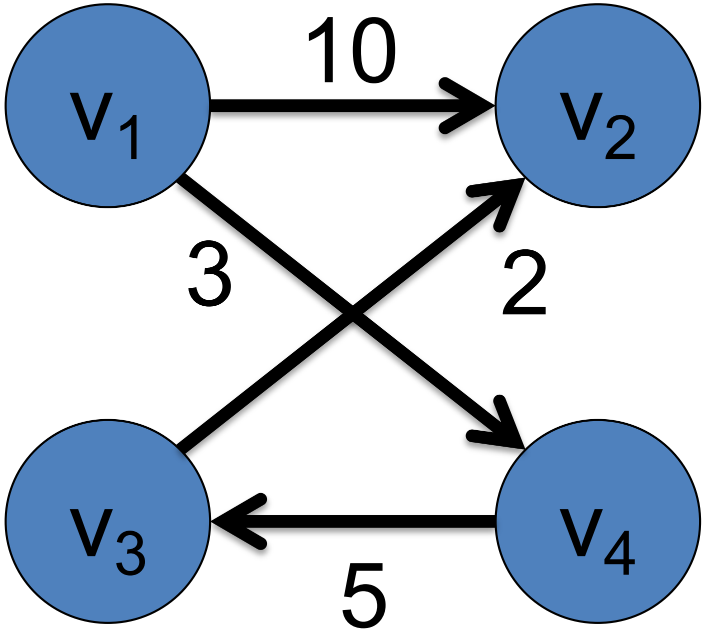
✓ $\{(v_1,v_2),(v_1,v_4),(v_3,v_2)\}$?

$\{(v_3,v_2)\}$?

$\{(v_1,v_4),(v_3,v_2)\}$?

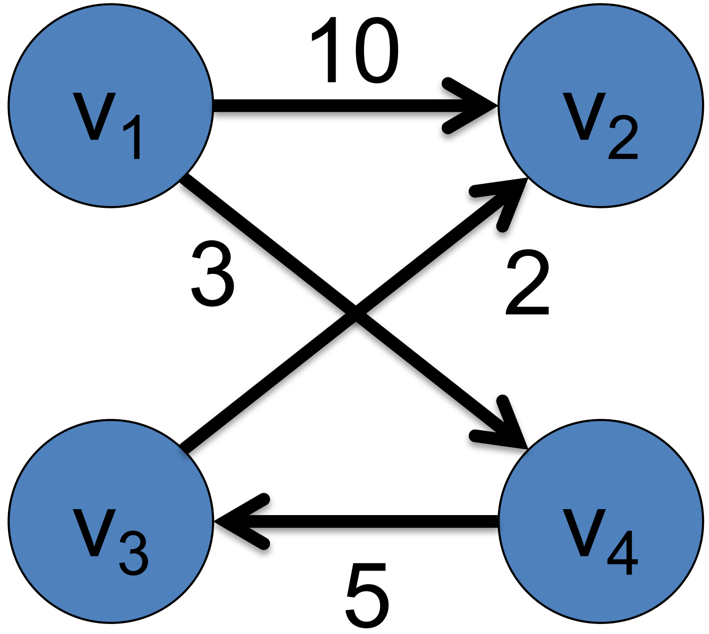
Cut

D = (V, A)



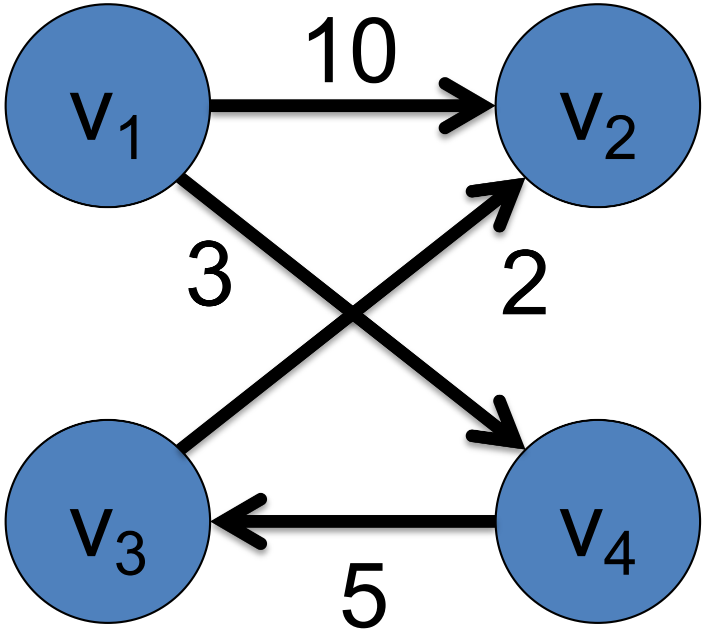
C = out-arcs(U)

Capacity of Cut



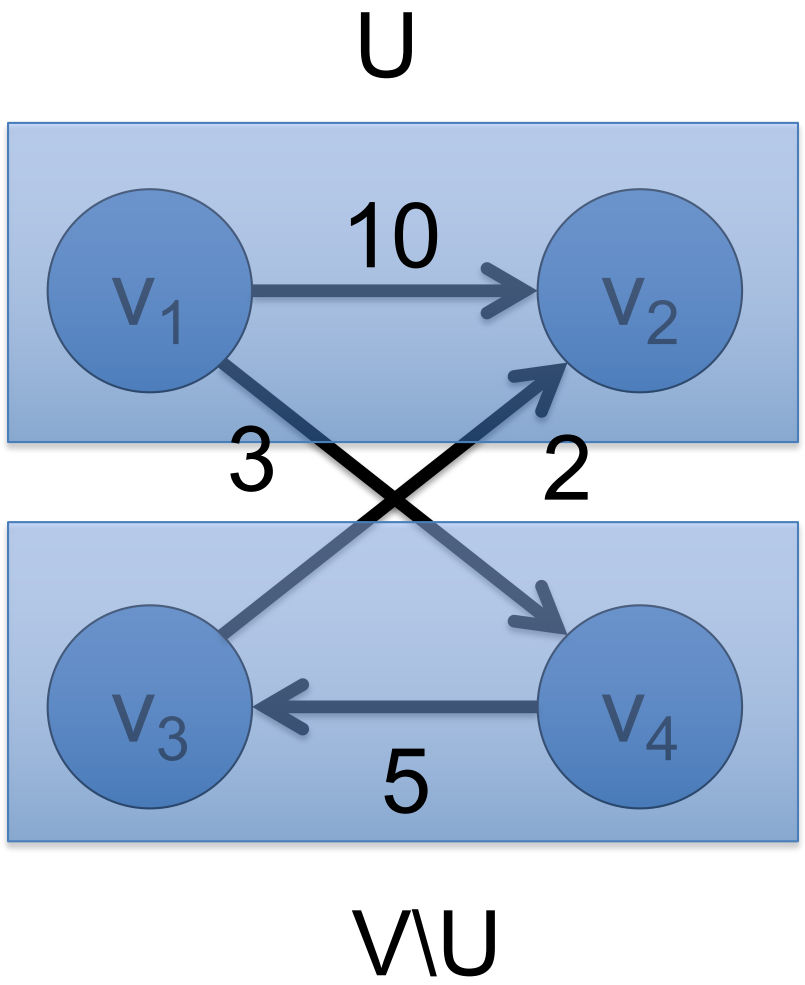
Sum of capacity of all arcs in C

Capacity of Cut



$$\Sigma_{a \in C}\, c(a)$$

Capacity of Cut



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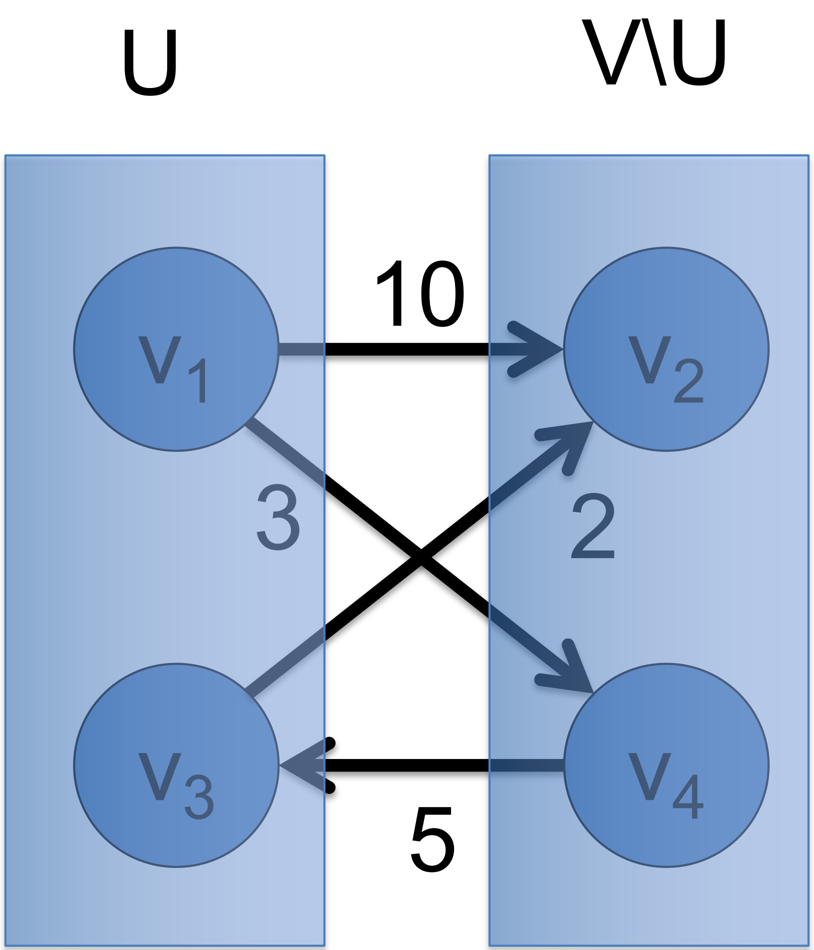
Capacity of Cut



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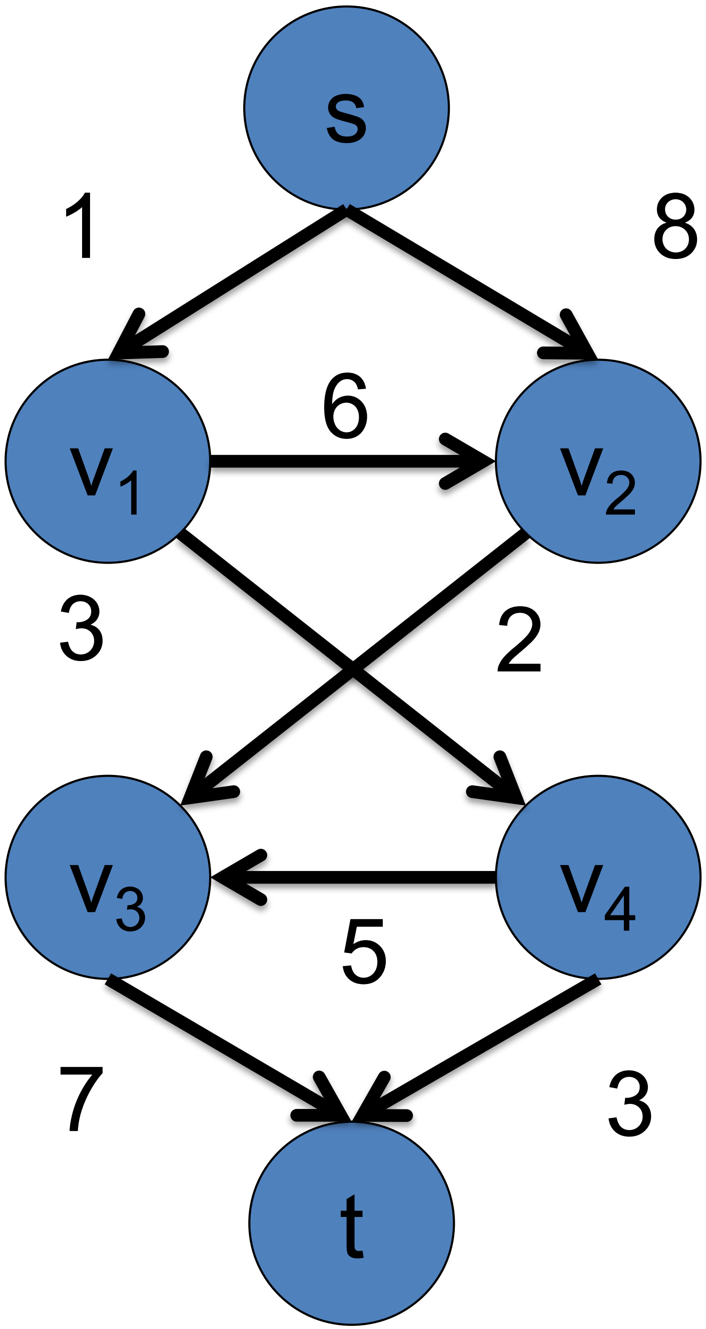
s-t Cut



D = (V, A)

A source vertex “s”

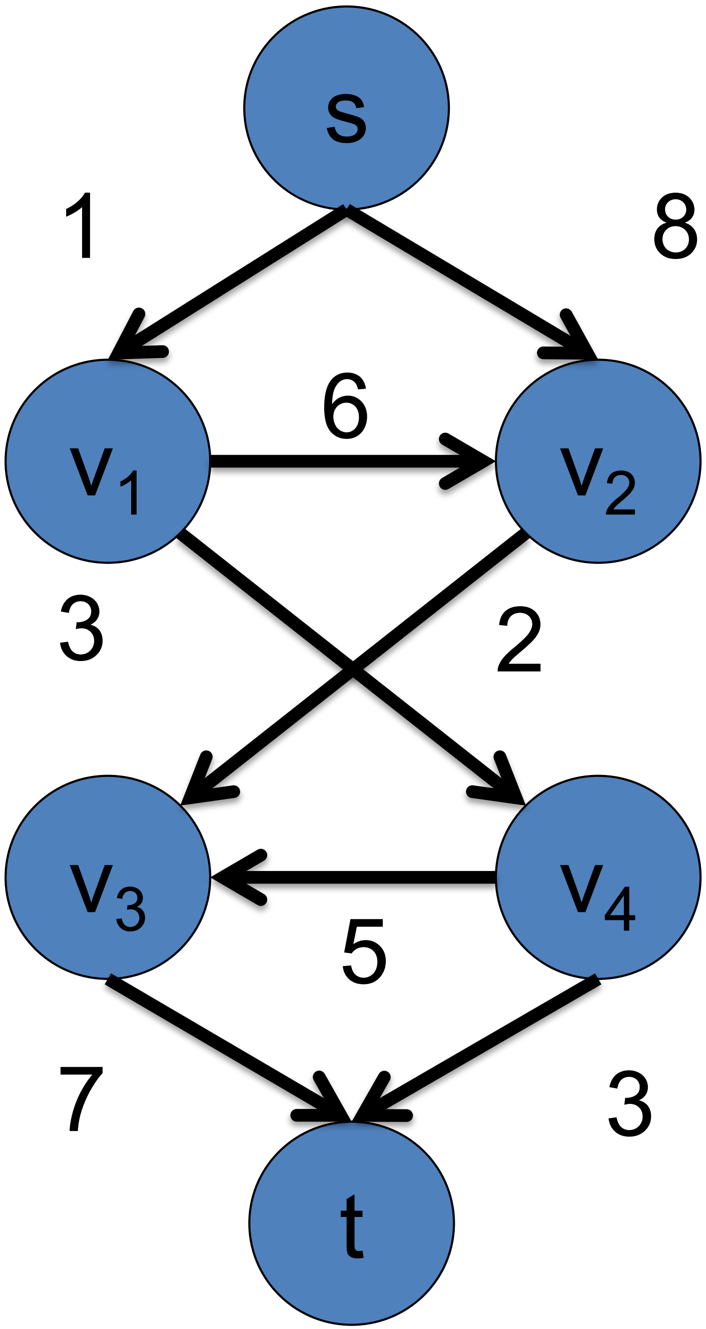
A sink vertex “t”

C is a cut such that

- $s \in U$
- $t \in V \backslash U$

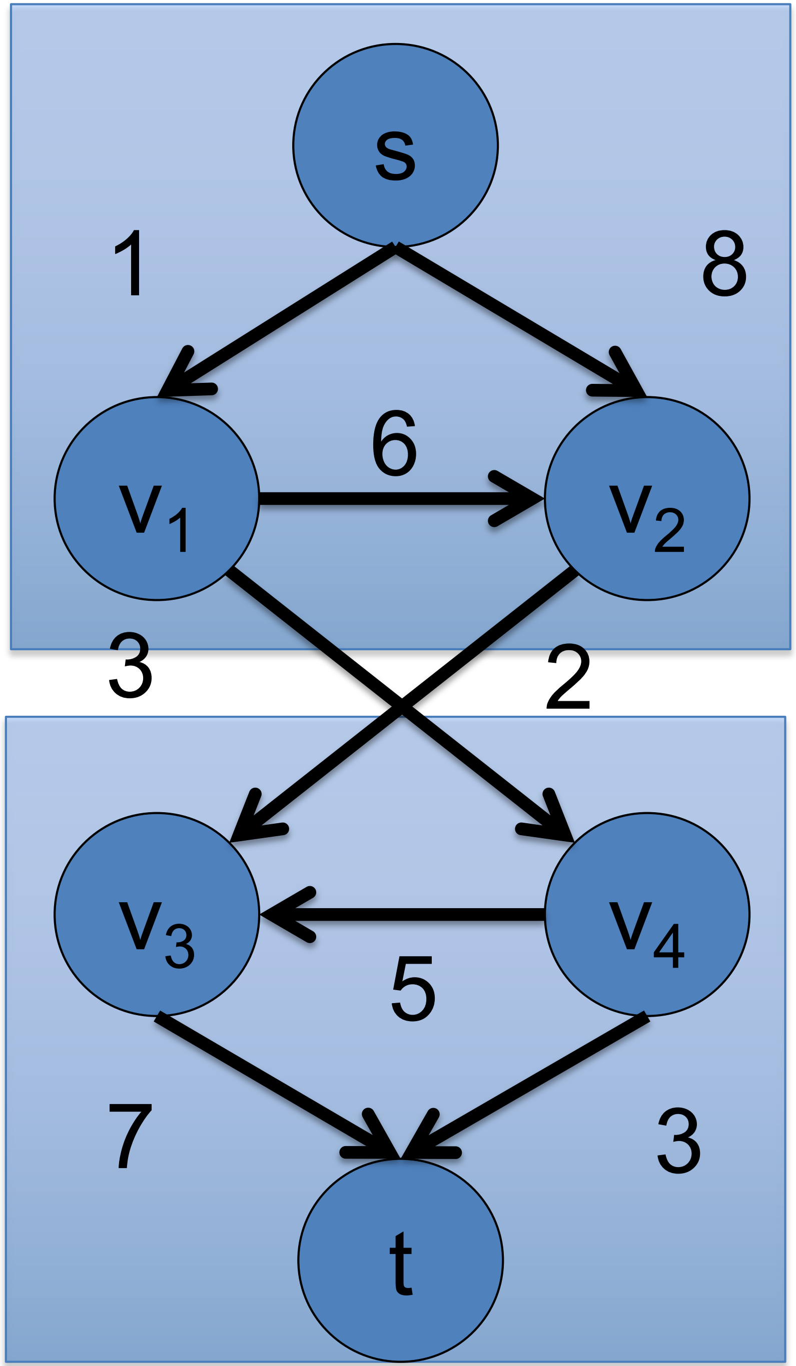
C is an s-t cut

Capacity of s-t Cut



$\Sigma_{a \in C}\ c(a)$

Capacity of s-t Cut



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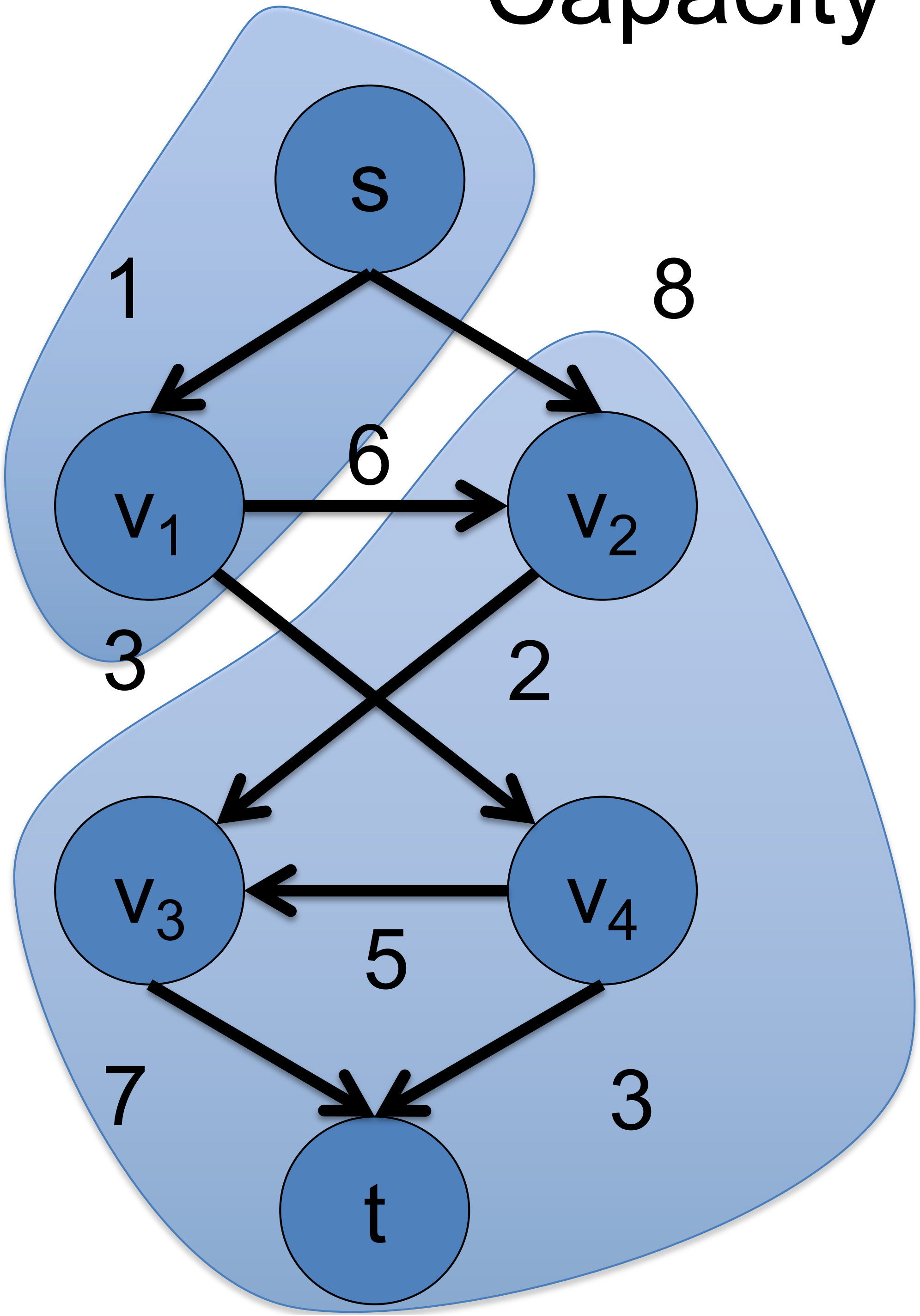
Capacity of s-t Cut



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