# Some linear programming methods for frontier 

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#### Abstract

We propose new methods for estimating the frontier of a set of points. The estimates are defined as kernel functions covering all the points and whose associated support is of smallest surface. They are written as linear combinations of kernel functions applied to the points of the sample. The weights of the linear combination are then computed by solving a linear programming problem. In the general case, the solution of the optimization problem is sparse, that is, only a few coefficients are non zero. The corresponding points play the role of support vectors in the statistical learning theory. In the case of uniform bivariate densities, the $L_{1}$ error between the estimated and the true frontiers is shown to be almost surely converging to zero, and the rate of convergence is provided. The behaviour of the estimates on one finite sample situation is illustrated on simulations.


Keywords: Functional estimate, Linear programming, $L_{1}$ error, frontier estimation.

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## 1 Introduction

Many proposals are given in the literature for estimating a set $S$ given a finite random set of points drawn from the interior. This problem of frontier or support estimation arises in classification (HaRDY \& Rasson [24]), clustering problems (Hartigan [25]), discriminant analysis (Baufays \& RasSON [3]), and outliers detection. Applications are found in medical diagnosis (Tarassenko et al [33]) as well as in condition monitoring of machines (Devroye \& Wise [12]). In image analysis, the segmentation problem can be considered under the support estimation point of view, where the support is a convex bounded set in $\mathbb{R}^{2}$ (Korostelev \& Tsybakov [30]). We also point out some applications in econometrics (e.g. Deprins, et al [11]). In such cases, the unknown support can be written

$$
\begin{equation*}
S=\{(x, y): 0 \leq x \leq 1 ; 0 \leq y \leq f(x)\}, \tag{1}
\end{equation*}
$$

where $f:[0,1] \rightarrow(0,+\infty)$ is an unknown function. Here, the problem reduces to estimating $f$, called the production frontier (see for instance Härdle et al [22]). The data consist of pair ( $X, Y$ ) where $X$ represents the input (labor, energy or capital) used to produce an output $Y$ in a given firm. In such a framework, the value $f(x)$ can be interpreted as the maximum level of output which is attainable for the level of input $x$. Korostelev et al [29] suppose $f$ to be increasing and concave, from economical considerations, which suggests an adapted estimator, called the DEA (Data Envelopment Analysis) estimator. It is the lowest concave monotone increasing function covering all the sample points. Therefore it is piecewise linear and, up to our knowledge, it is the first frontier estimate computed thanks to a linear programming technique (Charnes et al [8]). Its asymptotic distribution is established by Gijbels et al [16].

An early paper was written by Geffroy [14] for independent identically distributed observations from a density $\phi$. The proposed estimator is a kind of histogram based on the extreme values of the sample. This work was extended in two main directions.

On the one hand, piecewise polynomials estimates were introduced. They are defined locally on a given slice as the lowest polynomial of fixed degree covering all the points in the considered slice. Their optimality in an asymptotic minimax sense is proved under weak assumptions on the rate of decrease $\alpha$ of the density $\phi$ towards 0 by Korostelev \& Tsybakov [30] and by Härdle et al [23]. Extreme values methods are then proposed by Hall et al [21] and by Gijbels \& Peng [15] to estimate the parameter $\alpha$.

On the other hand, different propositions for smoothing Geffroy's estimate were made in the case of a Poisson point process. Girard \& Jacob [19] introduced estimates based on kernel regressions and orthogonal series method [17, 18]. In the same spirit, Gardes [13] proposed a Faber-Shauder estimate. Girard \& Menneteau [20] introduced a general framework for studying estimates of this type and generalized them to supports writing

$$
S=\{(x, y): x \in E ; 0 \leq y \leq f(x)\},
$$

where $f: E \rightarrow(0,+\infty)$ is an unknown function and $E$ an arbitrary set. In each case, the limit distribution of the estimator is established.

We also refer to Abbar [1] and Jacob \& Suquet [27] who used a similar smoothing approach, although their estimates are not based on the extreme values of the Poisson process.

The estimate proposed in this paper can be considered to belong to the intersect of these two directions. It is defined as a kernel estimate obtained by smoothing some selected points of the sample. These points are chosen automatically by solving a linear programming problem to obtain an estimate of the support covering all the points and with smallest surface. Its advantages are the following: it can be computed with standard optimization algorithms (see e.g. Bonnans et al [6], chapter 4), its smoothness is directly linked to the smoothness of the chosen kernel and it benefits from interesting theoretical properties. For instance, we prove that it is almost surely convergent for the $L_{1}$ norm. The estimate is defined in Section 2. Its theoretical properties are es-
tablished in Section 3. The behaviour of the estimate is illustrated in Section 4 on one finite sample situation. It is compared to a similar proposition found in Barron et al [2]. We conclude by describing our further work in Section 5. Proofs can be found in Bouchard el al [5].

## 2 Boundary estimates

### 2.1 A linear programming problem

Let all the random variables be defined on a probability space $(\Omega, \mathcal{F}, P)$.
The problem under consideration is to estimate an unknown positive function $f:[0,1] \rightarrow(0, \infty)$ on the basis of observations $Z_{N}=\left(X_{i}, Y_{i}\right)_{i=1, \ldots, N}$. The former represents an i.i.d. sequence with pairs $\left(X_{i}, Y_{i}\right)$ being uniformly distributed in the set $S$ defined as in (1). For the sake of simplicity, we consider in the following the extension of $f$ on all $\mathbb{R}$ by introducing $f(x)=0$ for all $x \notin[0,1]$. Letting

$$
C_{f} \triangleq \int_{0}^{1} f(u) d u=\int_{\mathbb{R}} f(u) d u
$$

each variable $X_{i}$ is distributed in [0,1] with p.d.f. $f(\cdot) / C_{f}$ while $Y_{i}$ has the uniform conditional distribution with respect to $X_{i}$ in the interval $\left[0, f\left(X_{i}\right)\right]$.

The considered estimate of the frontier is chosen from the family of functions:

$$
\left\{\begin{array}{l}
\widehat{f}_{N}(x)=\sum_{i=1}^{N} K_{h}\left(x-X_{i}\right) \alpha_{i}, \quad K_{h}(t)=h^{-1} K(t / h),  \tag{2}\\
\alpha_{i} \geq 0, \quad i=1, \ldots, N,
\end{array}\right.
$$

where $K$ is a kernel function $K: \mathbb{R} \rightarrow[0, \infty)$ integrating to one and with bandwidth $h>0$. Each coefficient $\alpha_{i}$ represents the importance of the point ( $X_{i}, Y_{i}$ ) in the estimation. In particular, if $\alpha_{i} \neq 0$, the corresponding point ( $X_{i}, Y_{i}$ ) can be called a support vector by analogy with Support Vector Machines (SVM). We refer to Cristianini \& Shawe-Taylor [10] for a review on this topic and to Schölkopf \& Smola [32], chapter 8, for examples of application of SVM to quantile estimation. The constraint $\alpha_{i} \geq 0$ for all $i=1, \ldots, N$ ensures that $\widehat{f}_{N}(x) \geq 0$ for all $x \in \mathbb{R}$ and prevents the estimator from being too irregular.

Let us remark that the surface of the estimated support is given by

$$
\int_{\mathbb{R}} \widehat{f}_{N}(x) d x=\sum_{i=1}^{N} \alpha_{i}
$$

This suggests to define the vector parameter $\alpha=\left(\alpha_{1}, \ldots, \alpha_{N}\right)^{T}$ from a linear program as follows

$$
\begin{equation*}
J_{P}^{*} \triangleq \min _{\alpha} \mathbf{1}^{T} \alpha \tag{3}
\end{equation*}
$$

subject to

$$
\begin{align*}
& A \alpha \geq Y  \tag{4}\\
& \alpha \geq 0 \tag{5}
\end{align*}
$$

The following notations have been introduced:

$$
\begin{aligned}
\mathbf{1} & \triangleq(1,1, \ldots, 1)^{T} \in \mathbb{R}^{N} \\
A & \triangleq\left\{K_{h}\left(X_{i}-X_{j}\right)\right\}_{i, j=1, \ldots, N} \\
Y & \triangleq\left(Y_{1}, \ldots, Y_{N}\right)^{T}
\end{aligned}
$$

Hence, $A \alpha=\left(\widehat{f}_{N}\left(X_{1}\right), \ldots, \widehat{f}_{N}\left(X_{N}\right)\right)^{T}$, and the vector constraint (4) means that $\widehat{f}_{N}\left(X_{i}\right) \geq Y_{i}, i=1, \ldots, N$. In other words, $\widehat{f}_{N}$ defines the kernel estimate of the support covering all the points and with smallest surface. In practice (see Section 4 for an illustration) the solution of the linear program is sparse in the sense that $n(\alpha)=\#\left\{\alpha_{i} \neq 0\right\}$ is small (for a properly selected value of $h$, see Paragraph 2.2) and thus the resulting estimate is fast to compute even for large samples.

Let us note that the above described estimator (2)-(5) might be derived as the Maximum Likelihood Estimate related to the approximation family (2). Indeed, the joint probability density function for observations $Z_{N}$ given parameter function $f(x)$ can be written

$$
p\left(Z_{N} \mid f\right)=\prod_{i=1}^{N} \frac{f\left(X_{i}\right)}{C_{f}} \cdot \frac{1}{f\left(X_{i}\right)} \mathbf{1}\left\{0 \leq Y_{i} \leq f\left(X_{i}\right)\right\}
$$

where $\mathbf{1}\{$.$\} is the indicator function. Moreover,$

$$
\left.C_{f}\right|_{f=\widehat{f}_{N}}=\sum_{i=1}^{N} \alpha_{i}
$$

and therefore, the Log-Likelihood function is

$$
L(\alpha) \triangleq \log p\left(Z_{N} \mid \widehat{f}_{N}\right)=-N \log \sum_{i=1}^{N} \alpha_{i}+\sum_{i=1}^{N} \log \mathbf{1}\left\{Y_{i} \leq \widehat{f}_{N}\left(X_{i}\right)\right\}
$$

and its maximization over the set of non-negative parameters $\alpha$ is equivalent to problem (3)-(5).

### 2.2 Choice of the bandwidth

The bandwidth parameter $h$ has to be properly selected. The asymptotic results presented in Section 3 (see Corollary 1 and 2) only provide an order of magnitude and thus are useless in practical situations. Assuming that $(X, Y)$ is uniformly distributed on $S$, we have two different estimations of $E[Y]$ :

- $\hat{m}_{1}=\frac{1}{N} \sum_{i=1}^{N} Y_{i}$
- $\hat{m}_{2}=\frac{1}{N} \sum_{i=1}^{N} \frac{\hat{f}_{N}\left(X_{i} ; h\right)}{2}$

These estimations are expected to be equal whenever the estimated frontier $\hat{f}_{N}$ is close to $f$. Hence, we propose to select the value $\hat{h}_{N}$ minimizing the quantity

$$
D(h)=\frac{1}{N}\left|\sum_{i=1}^{N} Y_{i}-\frac{1}{2} \sum_{i=1}^{N} \hat{f}_{N}\left(X_{i} ; h\right)\right| .
$$

This criterion is tested on simulations in section 4.

### 2.3 Comparison with other methods

Let us remark that other solutions for estimating $\alpha$ in (2) have already been proposed. Girard \& Menneteau [20] considered a partition $\left\{I_{r}: 1 \leq r \leq k\right\}$ of $[0,1]$, with $k \rightarrow \infty$. For all $1 \leq r \leq k$, they introduce

$$
D_{r}=\left\{(x, y): x \in I_{r}, 0 \leq y \leq f(x)\right\},
$$

the slice of $S$ built on $I_{r}, Y_{r}^{*}=\max \left\{Y_{i} ;\left(X_{i}, Y_{i}\right) \in D_{r}\right\}$, and the estimates

$$
\widehat{\alpha_{i}}=\left\lvert\, \begin{array}{ll}
\lambda\left(I_{r}\right) Y_{r}^{*} & \text { if } \exists r \in\{1, \ldots, k\} ; Y_{i}=Y_{r}^{*} \\
0 & \text { otherwise },
\end{array}\right.
$$

where $\lambda$ is the Lebesgue measure. They propose the following frontier estimate

$$
\check{f}_{N}(x)=\sum_{r=1}^{k} K_{h}\left(x-x_{r}\right) \lambda\left(I_{r}\right) Y_{r}^{*},
$$

where $x_{r}$ is the center of $I_{r}$. This approach suffers from a practical difficulty: the choice of the partition and more precisely the choice of $k$. In our context, solving the linear problem (3)-(5) direcly yields the support vectors, once $h$ has been chosen according to the method described in Paragraph 2.2.

In this sense, the estimate proposed in Barron et al [2] is similar to $\widehat{f}_{N}$. It is defined by the Fourier expansion:

$$
\widehat{g}_{N}(x)=c_{0}+\sum_{k=1}^{M} a_{k} \cos (2 \pi k x)+\sum_{k=1}^{M} b_{k} \sin (2 \pi k x),
$$

where the vector of parameters $\beta=\left(c_{0}, a_{1}, \ldots, a_{M}, b_{1}, \ldots, b_{M}\right)^{T}$ is solution of the linear programming problem:

$$
\begin{equation*}
\min c_{0} \quad\left(=\int_{0}^{1} \widehat{g}_{N}(x) d x\right) \tag{6}
\end{equation*}
$$

under the constraints

$$
\begin{align*}
\widehat{g}_{N}\left(X_{i}\right) & \geq Y_{i}, \quad i=1, \ldots, N  \tag{7}\\
\sum_{k=1}^{M} k\left(\left|a_{k}\right|+\left|b_{k}\right|\right) & \leq L /(2 \pi) \tag{8}
\end{align*}
$$

Therefore, $\widehat{g}_{N}$ defines the Fourier estimate of the support covering all the points (equation (7)), $L$-Lipschitzian (equation (8)) and with smallest surface (equation (6)). From the theoretical point of view, this estimate benefits from minimax optimality. It is compared to $\widehat{f}_{N}$ on one practical situation in Section 4 for different choices of parameters $M, L$ and $h$.

## 3 Theoretical results

In this section, we establish that $\widehat{f}_{N}$ is almost surely convergent for the $L_{1}$ norm on $[0,1]$. To this end, the basic assumptions on the unknown boundary function are introduced:

A1. $0<f_{\min } \leq f(x)<f_{\max }<\infty$, for all $x \in[0,1]$,
A2. $|f(x)-f(y)| \leq L_{f}|x-y|$, for all $x, y \in[0,1], \quad L_{f}<\infty$.

The following assumptions on the kernel function are considered:
B1. $K(t)=K(-t) \geq 0$,
B2. $\int K(t) d t=1$,
B3. $|K(s)-K(t)| \leq L_{K}|s-t|, L_{K}<\infty$,
B4. $\int K^{2}(t) d t<\infty$ and $\int t^{2} K(t) d t<\infty$.
In the following, we note $\left\|\hat{f}_{N}-f\right\|_{1}=\int\left|f(x)_{N}-f(x)\right| d x$.
Theorem 1 Let $h \rightarrow 0$ and $\log N /\left(N h^{2}\right) \rightarrow 0$ as $N \rightarrow \infty$. Let the above mentioned assumptions $A$ and $B$ hold true. Then, estimator (2)-(5) has the following asymptotic properties:

$$
\limsup _{N \rightarrow \infty} \varepsilon_{1}^{-1}(N)\left\|\hat{f}_{N}-f\right\|_{1} \leq C<\infty \quad \text { a.s. }
$$

with $\varepsilon_{1}(N) \triangleq \max \left\{h, \sqrt{\log N /\left(N h^{2}\right)}\right\}$.

Corollary 1 The maximum rate of convergence which is guaranteed by Theorem 1

$$
\left\|\hat{f}_{N}-f\right\|_{1}=O_{p}\left((\log N / N)^{1 / 4}\right)
$$

is attained for $h \asymp(\log N / N)^{1 / 4}$.

This rate of convergence can be ameliorated at the price of a slight modification of the estimate. In the following, an additional constraint is considered in order to impose to each coefficient $\alpha_{i}$ to be of order $1 / N$. The counterpart of this modification is that the new estimate $\tilde{f}_{N}$ will usually rely on more support vectors than $\widehat{f}_{N}$.

Let us modify the estimator (2)-(5) as follows.

$$
\begin{equation*}
\tilde{f}_{N}(x)=\sum_{i=1}^{N} K_{h}\left(x-X_{i}\right) \alpha_{i} \tag{9}
\end{equation*}
$$

where vector $\alpha=\left(\alpha_{1}, \ldots, \alpha_{N}\right)^{T}$ is defined from the Modified Linear Program

$$
\begin{equation*}
J_{M P}^{*} \triangleq \min _{\alpha} \mathbf{1}^{T} \alpha \tag{10}
\end{equation*}
$$

subject to

$$
\begin{align*}
& A \alpha \geq Y  \tag{11}\\
& 0 \leq \alpha \leq C_{\alpha} / N \tag{12}
\end{align*}
$$

with a constant

$$
\begin{equation*}
C_{\alpha}>f_{\max } \tag{13}
\end{equation*}
$$

Remark. In fact, we need to ensure $C_{\alpha}>C_{f}$ which is implied by (13).

The modified estimator (9)-(13) differs from that of (2)-(5) by additionally bounding each $\alpha_{i}$ from above, see constraints (12).

Theorem 2 Let $h \rightarrow 0$ and $\log N /(N h) \rightarrow 0$ as $N \rightarrow \infty$. Let kernel function $K(\cdot)$ has a finite support, that is $K(t)=0 \forall|t| \geq 1$, and the assumptions $A$ and $B$ hold true. Then, estimator (9)-(13) has the following asymptotic properties:

$$
\begin{aligned}
& \limsup _{N \rightarrow \infty} \varepsilon_{2}^{-1}(N)\left\|\tilde{f}_{N}-f\right\|_{1} \leq C<\infty \quad \text { a.s. } \\
& \text { with } \varepsilon_{2}(N) \triangleq \max \{h, \sqrt{\log N /(N h)}\} \text {. }
\end{aligned}
$$

Remark. The support of $K(\cdot)$ is fixed to be the interval $[-1,1]$ without loss of generality.

Corollary 2 The maximum rate of convergence which is guaranteed by Theorem 2

$$
\left\|\tilde{f}_{N}-f\right\|_{1}=O_{p}\left((\log N / N)^{1 / 3}\right)
$$

is attained for $h \asymp(\log N / N)^{1 / 3}$.

Note that the rates of convergence of the DEA and FDH estimators are $O_{p}\left(N^{-2 / 3}\right)$ (see Kneip et al [28]) and $O_{p}\left(N^{-1 / 2}\right)$ (see Park et al [31]), but stronger assumptions are required (monotonicity or concavity of the frontier function $f$ ).


Figure 1: Illustration of the kernel estimation on a simulation.

## 4 Numerical experiments

The simulations presented here illustrate the behaviour of the kernel estimator $\widehat{f}_{N}$ compared to the estimator based on Fourier expansions $\widehat{g}_{N}$ proposed in Barron et al [2]. Since the Fourier estimate $\widehat{g}_{N}$ requires the unknown function to be periodic, we choose $f$ such that $f(0)=f(1)$. Besides, to avoid boundary effects on the input domain, we consider functions that are nearly zero when $x$ is close to 0 or 1 . In more general situations, boundary corrections should be implemented (see Cowling \& Hall [9]). The chosen function

$$
\begin{aligned}
f(x)=0.1 & +5(x-0.1) \mathbf{1}_{\{x>0.1\}} \\
& -5(x-0.2) \mathbf{1}_{\{x>0.2\}} \\
& +1(x-0.5) \mathbf{1}_{\{x>0.5\}} \\
& -9(x-0.8) \mathbf{1}_{\{x>0.8\}} \\
& +8(x-0.9) \mathbf{1}_{\{x>0.9\}},
\end{aligned}
$$

is piecewise linear and locally Lipschitizian with a Lipschitz constant $L_{f}=8$. Its graph is given on the right part of Figure 1 (dotted line). Here, $N=50$ points are uniformly generated on the domain $S$ upper bounded by $f$ (left part of Figure 1). The smoothing parameter $h$ is chosen according to Paragraph 2.2. On the right, the kernel estimate (continuous line) is superimposed to the unknown function $f$. The squares represent the points for which $\widehat{f}_{N}\left(X_{i}\right)=Y_{i}$. The triangles represent the support vectors (i.e. the points for which $\alpha_{i}>0$ ). On this example, the estimate $\widehat{g}_{N}$ is not visually different from $\widehat{f}_{N}$.

Thus, a more precise comparison should be done. For each estimate, the $L_{1}$
error $\Delta_{N}$ as well as the number of effective parameters $n p$ (that is $n_{\alpha}$ and $n_{\beta}=$ $\left.\#\left\{\beta_{i} \neq 0\right\}\right)$ are evaluated for $N=25$ and $N=100$. The average value and the standard deviation of these quantities are computed on 1000 replications. The estimation is carried out with different values of the parameters, namely $h$ for the kernel estimate, and $L$ and $M$ for the Fourier estimate. For each value of $h$ the criterion $D(h)$ presented in Paragraph 2.2 is evaluated. The adaptive choice of $L$ and $M$ parameters is not implemented in this setting. The results are summarized in Tables 1 and 2. The lowest error is emphasized for each estimate. It can be noted that the mean $L_{1}$ error of both estimates are very similar. In fact, the kernel estimate seems to give a slight lower error for small number of points and the Fourier estimate yields better results for large sample size situations, confirming its asymptotic optimality. Let us note that the standard deviation of the $L_{1}$ error is in general smaller for the kernel estimate. Regarding the number of parameters, the kernel estimate seems to be more parsimonious than the Fourier estimate. Finally, we can notice that our criterion $D(h)$ yields a reasonable choice of $h$, although a slight oversmoothing appears for small sample sizes.

## 5 Further work

As noticed in Section 3, the rate of convergence of the $L_{1}$ error is not optimal. This problem can be overcomed by modifying the constraints of the linear programming problem in order to impose to the estimate the same Lipschitz properties as the unknown function $f$. We also plan to adapt our proofs to more general settings, namely non uniform distributions with multidimensional support. The extension to the case where $X$ is multidimensional is straightforward since it suffices to consider multivariate kernels. The non-uniform case requires assumptions on the behaviour of the distribution in the unknown boundary neighborhood. From the practical point of view, the selection procedure of the smoothing parameter has to be improved.

| estimate | $h \quad L$ |  | $\Delta_{N}$ | np | $D(h)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| kernel | 0.120 |  | 0.113 (0.033) | 4.476 (0.840) | 43.692 |
|  | 0.150 |  | 0.111 (0.032) | 3.613 (0.649) | 38.232 |
|  | 0.180 |  | 0.122 (0.027) | 3.093 (0.623) | 34.362 |
|  | 0.210 |  | 0.137 (0.023) | 2.702 (0.590) | 32.258 |
|  | 0.240 |  | 0.153 (0.023) | 2.434 (0.557) | 33.793 |
|  | 0.270 |  | 0.163 (0.028) | 2.184 (0.403) | 37.642 |
| Fourier | 3 | 4 | 0.142 (0.036) | 4.534 (0.741) |  |
|  | 5 | 4 | 0.116 (0.043) | 5.532 (1.008) |  |
|  | 7 | 4 | 0.124 (0.040) | 6.636 (1.193) |  |
|  | 9 | 4 | 0.135 (0.040) | 7.312 (1.235) |  |
|  | 11 | 4 | 0.143 (0.041) | 7.690 (1.219) |  |
|  | 13 | 4 | 0.150 (0.041) | 7.870 (1.172) |  |
| Fourier | 3 | 8 | 0.143 (0.036) | 4.545 (0.757) |  |
|  | 5 | 8 | 0.117 (0.043) | 5.574 (1.055) |  |
|  | 7 | 8 | 0.126 (0.041) | 6.771 (1.321) |  |
|  | 9 | 8 | 0.139 (0.042) | 7.691 (1.584) |  |
|  | 11 | 8 | 0.150 (0.042) | 8.376 (1.756) |  |
|  | 13 | 8 | 0.159 (0.042) | 8.877 (1.900) |  |

Table 1: Results for 1000 simulations with $N=25$ points. The mean value of $\Delta_{N}$ and $n p$ is given with the standard deviation in parentheses.

| estimate | $h \quad L$ | M | $\Delta_{N}$ | np | $D(h)$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| kernel | 0.050 |  | 0.072 (0.015) | 13.777 (1.351) | 31.972 |
|  | 0.070 |  | 0.058 (0.013) | 9.976 (1.249) | 23.200 |
|  | 0.090 |  | 0.058 (0.011) | 7.558 (1.104) | 17.611 |
|  | 0.110 |  | 0.061 (0.011) | 5.696 (0.887) | 15.210 |
|  | 0.130 |  | 0.073 (0.011) | 4.725 (0.714) | 15.881 |
|  | 0.150 |  | 0.082 (0.012) | 3.964 (0.524) | 17.337 |
| Fourier | 3 | 4 | 0.123 (0.021) | 5.080 (0.716) |  |
|  |  | 4 | 0.073 (0.019) | 5.792 (0.781) |  |
|  | 7 | 4 | 0.060 (0.011) | 7.850 (0.979) |  |
|  | 9 | 4 | 0.063 (0.012) | 8.751 (0.579) |  |
|  | 11 | 4 | 0.067 (0.014) | 8.896 (0.362) |  |
|  | 13 | 4 | 0.069 (0.015) | 8.955 (0.235) |  |
| Fourier | 3 | 8 | 0.124 (0.021) | 5.117 (0.752) |  |
|  |  | 8 | 0.073 (0.020) | 5.889 (0.863) |  |
|  | 7 | 8 | 0.057 (0.012) | 8.315 (1.470) |  |
|  | 9 | 8 | 0.057 (0.013) | 10.620 (1.672) |  |
|  | 11 | 8 | 0.061 (0.014) | 12.454 (1.858) |  |
|  | 13 | 8 | 0.067 (0.014) | 13.890 ( 1.879) |  |

Table 2: Results for 1000 simulations with $N=100$ points. The mean value of $\Delta_{N}$ and $n p$ is given with the standard deviation in parentheses.

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