A Scale-space Analysis of Multiplicative Texture Processes

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***Abstract*— Gaussian Scale-space describes the local structure of images. This paper shows a stochastic analysis of the diffusion equation as put forward by Koenderink (1984) for regular images. Important classes of the stochastic process which are structurally described by the diffusion analysis include Brownian fractals, Markovian textures, and fragmentation processes. The analysis shows the diffusion coefficient to relate to the local autocorrelation function over the diffusion process. Diffusion of the multiplicative image formation process directly leads to power-law statistics over the diffusion scale, and to Weibull statistics in the spatial domain.**

I. Introduction

The observation of images is governed by a long-term scaling regime, which diffuses the scene irradiance to a resolution limited sensory observation. The statistics of the imaging process is governed by a enormous scale range. On the quantum scale, the stochastic process of photon scattering determines spatial interactions. On a microscopic scale, the rough surface of materials cause multiple scattering, blocking, and vignetting. At meso-scale, long-range correlations are imposed by shadow and shading effects, object borders, and interreflections. Finally, at a macro-scale, occlusion and clutter dominate image statistics.

In all these cases, I consider the spatio-spectral energy distribution impinging on an image sensor to be the result of a stochastic process. Hence, the image *before* observation may be characterized by the probability density describing the random nature of the energy fluctuations, and the correlation function describing how a localized fluctuation influences the local, regional, or total energy density. The observation may be considered to continue this correlation function under a Gaussian diffusion regime.

Note that the diffusion process is not only governed by the sensory system, that is the lens system, photoreceptor sensitivity and dimensions, and photoreceptor spacing. Besides these diffusion effects imposed by a resolution limited imaging system, the interaction between light and matter causes diffuse reflectance. The diffusion process is formalized by the Kubelka-Munk equations [1], [2]. Furthermore, interaction between reflected light, and the often non-homogeneous medium it travels through, causes diffusion captured by the Lambert-Beer law [3]. These diffusion processes are not necessarily linear, and interference effects certainly result in a non-causal relation between the energy density as falling on the imaging sensor and the original material surface.

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An important non-linear effect introduced by the observation of a three-dimensional world is the effect of occlusion and shadow. Occlusion is the event that an imaged object is partly invisible due to hiding by another object which is in the field of view. Shadow is the effect caused by the light source being held from irradiating an object due to blocking of another object. Both effects introduce singularities in the bidirectional reflectance distribution function (BRDF). As these effects will be present at any scale, the BRDF consists of infinite many discontinuities [4]. Observation of such a typical BRDF may have implication beyond the causal Gaussian observations [5]. The diffusion processes before the retina will be reflected in the statistics of the observation process.

Consider the observation of a surface at macro-scale, for which several orders of magnitude have to be covered by the diffusing process to transform the photon-scale characteristics into surface reflection properties. On the other hand, the scale difference between one objects and a few objects interacting to cause occlusion and shadow effects, is typically covered by a small scale step. Both scaling processes are typically present in a general image. Hence, any statistical theory to describe natural image statistics should consider several orders of magnitude in scale. In this respect, fractals are long range scale idealizations [6], representing the result over a long diffusion regime.

In this paper, I consider the scaling behavior of multiplicative stochastic processes. Hence, the processes as resulting from a smoothly varying illumination flux modified by a multiplicative, high frequent, stochastic reflectance function. Multiplicative image formation process is statistically modelled and shown to result in a power-law correlation function when observed at a finite scale. The scaling behaviour of the stochastic process is shown to relate the diffusion coefficient to the autocorrelation function over the diffusion process.

II. Image Formation and Power-law Statistics

Consider an infinitesimal small surface patch of homogeneous reflectance. When illuminated by incident light with spectral distribution $e(\lambda)$, light scattering within the material causes diffuse body reflection, while Fresnel interface reflectance occurs at the surface boundaries. I consider the Kubelka-Munk theory as a general model for color image formation, for simplicity formulated in one dimension:

$$E(\lambda, x) = \Phi(x)e(\lambda, x)\left(1-\rho(x)\right)^2 R(\lambda, x) + \Phi(x)e(\lambda, x)\rho(x) \tag{1}$$

where x denotes the position at the imaging plane and λ the wavelength. Further, $e(\lambda, x)$ denotes the illumination spectrum, $\rho(x)$ the Fresnel reflectance at x, and $\Phi(x)$ the flux of the light due to the angle between surface normal and light source direction. The surface albedo function is defined by $R(\lambda, x)$. The Fresnel reflectance is dominant at surface points for which the viewing direction coincides with the light source direction. This unlikely event will not dominate image formation in general, although it may have consequences for specific images. Hence, Lambertian reflection dominates image formation, in which case $\rho(x) \approx 0$. Note that the restriction is easily eliminated later on.

In a complex scene as natural images, material patches will often not be directly illuminated by sunlight. Diffusion of the sunlight due to atmospheric circumstances causes a diffuse light flux. Furthermore, light will be affected by multiple reflections on macro-scale objects, and inter-reflections due to the microscopic surface roughness. All these effects interact on the light, that is, a *fraction* of the light is reflected towards the surface patch which is imaged. Hence, if I consider a single ray of light, the intensity of the light finally reaching the camera is the result of several multiplicative processes,

$$E(\lambda, x) = \alpha_1(\lambda, x)\alpha_2(\lambda, x)\ldots\alpha_n(\lambda, x)e(\lambda, x) \qquad (2)$$

the factors $0 < \alpha_i < 1$ denoting the fraction of light reflected at each interaction.

Furthermore, note that an infinitesimal surface patch will not be illuminated by a single lightray, rather by a plurality of lightrays, each affected by various multiplicative processes. Hence, light from a surface patch imaged by a camera sensor is the effective result of a superposition of various multiplicative processes. As a consequence, there is a lower bound on the reflected intensity, larger than zero, as always some light will reach the material surface. Furthermore, materials can not be completely absorbing, and always some Fresnel reflectance will be present. Hence, no truly "black" exists, as some light will be reflected. An upper bound on the reflected intensity is given by the fact that in general no light is added to the illumination intensity. Hence $\alpha_i < 1$ results in a converging process towards zero. These boundaries turn out to impose a dramatic effect on image statistics.

The whole process is of a stochastic nature, i.e. the individual α_i are the results of coincidental interactions between various materials causing reflection and absorptions. Where the diffuse lightsource $e(\lambda, x)$ is a slowly varying function of x, the α_i are high frequency variations caused by material roughness, interreflections, occlusions, and masking. As proven by Levy and Solomon, the boundaries on the multiplicative process impose the constrained converging multiplicative process to lead to a power-law distribution in the resulting variable $E(.)$ [7].

Following [8], the proof is simply obtained by applying a log transform on (eq. 2), $l_i = \log \alpha_i$ and $y = \log E$, and rewriting (eq. 2) in a recurrent relation

$$y_{i+1} = y_i + l_i \ . \qquad (3)$$

In the transformed domain, the process describes a random walk with a drift $\langle l \rangle < 0$. The lower boundary ensures convergence of the process rather than escape to $-\infty$. The process is described by the master equation [7]

$$P(y, i+1) = \int_{-\infty}^{\infty} \pi(l) P(y-l, i) dl \ , \qquad (4)$$

where $\pi(l)$ denotes the transformed distribution of the original probability density Π of α_i. Exact solution of the master equation is obtained by considering the maximum value y_{max} reached by the random walk over all times. The problem has been solved [9] using renewal theory. In summary, a superposition of random walkers establish a uniform flux directed towards $-\infty$. The density of these walkers in the positive direction is decaying, given by a Wiener-Hopf integral [8] of which the solution is

$$P_{max}(\max(0, y_{max}))\ e^{-\mu y_{max}} \ , \qquad (5)$$

with μ given by

$$\int_0^{\infty} \Pi(\alpha)\alpha^{\mu} d\alpha = 1 \ . \qquad (6)$$

Hence, the power-law distribution is controlled by the extreme values of the diffusion process. Substituting the original variables for the transformations y, l yields a power-law,

$$P(E) = cE^{-\mu} \ , \qquad (7)$$

c representing a scaling constant. Details on the derivation are given by Sornette and Cont [8].

An alternative approach is to approximate the master equation by a Fokker-Planck equation,

$$\frac{\partial P(y,i)}{\partial i} = -v\frac{\partial P(y,i)}{\partial y} + D\frac{\partial^2 P(y,i)}{\partial y^2} \ , \qquad (8)$$

where $v = \langle l \rangle$ and $D = \langle l^2 \rangle - \langle l \rangle^2$. The details of π are not important for large i. The Langevin equation of the process is approximated by

$$\frac{dy}{di} = v + \eta(i) \ , \qquad (9)$$

which exemplifies the competition between drift v and diffusion $\eta(i)$.

Note that a stochastic variable of the form $w_{i+1} = b_i + a_i w_i$, with a_i and b_i inducing random fluctuations, also leads to a power-law [8]. Hence, we may eliminate our assumption of pure Lambertian reflection, yielding the derived results to hold in general image formation.

Note that the exact probability distribution π from which the α_i are drawn is not of importance. The process arises as a consequence of the central limit theorem in the log-transformed domain. Hence, power-law behavior for multiplicative processes is as natural as Boltzmann laws for additive fluctuations [7].

III. Multi-scale Observation and Multiplicative Fluctuations

Image formation is a multiplicative process, covered by power-law behavior if a large number of fluctuations is applied. Observation of an image is well known to be covered by linear correlation between a sensor of limited resolution and the intensity of the light. The convolution operator averages over the multiplicative stochastic process. Hence, it is not directly clear when the process is still dominated by multiplicative fluctuations, and when the additive nature of convolution takes over. Intuitively, the multiplicative process will dominate as long as spatial observations are correlated, that is, the spatial distribution is affected by one source of multiplicative fluctuations. In that case, the weighted sum reflects the multiplicative nature of the fluctuations. When the spatial extent is such that large parts of the sensor area are uncorrelated, the observation may be considered as the summation of independent stochastic processes. The latter case will result in Gaussian statistics, due to the central limit theorem.

To explain this trade-off, consider the observation of a multiplicative process at two scales a small distance apart. Observation of the multiplicative stochastic process by a sensor of limited size implies correlation of the signal with the sensor sensitivity function, the sensor being limited in resolution due to some spatial extent t. The scaling behavior of the resulting measurement is examined when changing the spatial extent t to $t + \Delta t$, in which case

$$\hat{E}_{t+\Delta t} = a\alpha e_t + b\beta e_{\Delta t} \ . \tag{10}$$

Here, e_t refers to the diffuse light flux falling onto the surface patch imaged by the sensor of original size t, and $e_{\Delta t}$ the additional surface area covered by the increase in sensor size Δt. The factors a and b denote a simplified view on convolution, and represent spatial weighing terms. The factors α and β represent the stochastic multiplicative variables due to various light reflections from the surface area. The new measurement at decreased resolution is composed of the superposition of multiplicative process, which again can be proven to follow a power-law statistic [8]. The result is independent of the sensor sensitivity as long as the stochastic reflectance process dominates statistics rather than the deterministic process of convolution.

IV. Spatial Diffusion of a Power-law Correlated Process

So far I modelled the stochastic process of intensity fluctuations, driven by a multiplicative process, without considering large spatial extent. Here I will follow Mannella, Grigolini, and West [6] to introduce diffusion of a power-law correlated process. Hence, formalizing the observation process as intuitively sketched in the preceding paragraph.

Assume a variable $x(t)$ which undergoes a motion induced by a random variable $\epsilon(t)$,

$$\frac{dx}{dt} = \epsilon \ . \tag{11}$$

A single trajectory is given by time integration,

$$x(t) = \int_0^t \epsilon(t')dt' + x(0) \ . \tag{12}$$

As the stochastic behavior of $x(t)$ is of interest rather than single trajectories, the mean values of the moments of $x(t)$ are of importance. Furthermore, assume $\epsilon(t)$ fluctuates around zero, thus the first moment $\langle x(t) \rangle = \langle x(0) \rangle$. The second moment is given by

$$\langle x^2(t) \rangle = \int_0^t \int_0^t \langle \epsilon(t')\epsilon(t'') \rangle \, dt'' \, dt' + 2\int_0^t \langle \epsilon(t')x(0) \rangle \, dt' + \langle x^2(0) \rangle \ . \tag{13}$$

Assuming no correlation between $\epsilon(t)$ and the initial value $x(0)$, the second integral in (eq. 13) vanishes. Furthermore, by assuming the random process $\epsilon(t)$ to be stationary, the motion equation yields [6]

$$\frac{d}{dt} \langle x^2(t) \rangle = 2\int_0^t \langle \epsilon(\tau)\epsilon(0) \rangle \, d\tau \tag{14}$$

So far the only assumptions made are independence between $x(0)$ and ϵ, and stationarity of ϵ. When also the assumption of second-order stationarity is made, that is, ϵ is approximated by its second-order statistics. In this case, the statistical process x is Gaussian. Now consider an image as an integration of the total distribution ρ over all degrees of freedom Γ that are not of interest (see [6] for details),

$$E(x,t) = \int \rho(x, \epsilon\Gamma) d\epsilon d\Gamma \ . \tag{15}$$

After some manipulation, one obtains the final diffusion equation

$$\frac{\partial E(x,t)}{\partial t} = \alpha(t) \frac{\partial^2 E(x,t)}{\partial x^2} \ . \tag{16}$$

Note that (eq. 16) is equivalent to (eq. 14), i.e. multiplication by x^2 and integrating over x yields (eq. 14).

The derived diffusion equation (eq. 16) is equivalent to the result derived by Koenderink [5]. However, the function $\alpha(t)$ is defined by the Green-Kubo formula (eq. 14)

$$\alpha(t) = \int_0^t \langle \epsilon(\tau)\epsilon(0) \rangle \, d\tau \ , \tag{17}$$

which relates the diffusion constant to the autocorrelation function of the random fluctuations. This yields a direct relation between the local detail scale and the power-law autocorrelation function of the multiplicative process (eq. 7). Hence, the time axis may be rescaled,

$$t^* = t^{2-\mu} \ , \tag{18}$$

which, after some manipulations [6], leads to the Gaussian kernel

$$G(x,t;\mu) = \frac{1}{\sqrt{2\pi} t^{1-\mu/2}} \exp\left(-\frac{1}{2} \frac{x^2}{t^{2-\mu}} \right) \ . \tag{19}$$

Hence, scale-space sampling of a stochastic process of many uncorrelated trajectories lead to a Boltzmann law with exponential decay, for which logarithmic scale sampling has to be considered. In the case of correlated trajectories, the logarithmic sampling depends linearly on the fractal dimension $H_D = 1 - \mu/2$ of the process, hence leads to a finer sampling interval.

V. Consequence for Natural Image Statistics

Smoothing images by any filter kernel, including observation by a discrete sensory system, to obtain some multi-resolution representation will generate a power-law correlation function between the original image and its smoothed versions as function of filter size. The theoretical considerations sketched in Section II predict the results as empirically derived from large natural image collections [10], [11], [12], [13], [14].

The statistical characteristics of any spatial difference filter (zero-average filter) is indicative for the correlation function over scale [15], [16]. Consider a spatial difference filter, which may be decomposed in a derivative operator and a smoothing operator [5], [17],

$$h(x) = H(x) * \frac{\partial h}{\partial x} \ , \tag{20}$$

where $H(x)$ denotes the smoothing part of filter $h(x)$. The spatial derivative may be considered to act on an infinitesimal scale, whereas the smoothing operator covers the pixel size and filter scale. Within the infinitesimal scale, the response of the derivative operator will approximate the square root of the autocorrelation function over the diffusion scale. Hence, responses r for filter h applied to natural images will be power-law distributed. Averaging over a spatial extent implies the integration over a number of power-laws [18], yielding a probability distribution of filter responses

$$P(r) = e^{-\frac{1}{\gamma}\left|\frac{r}{\beta}\right|^{\gamma}} \ , \tag{21}$$

which is closely related to the Weibull distribution [19], [20]. In this equation, γ represents the Weibull shape parameter, and β the width of the distribution.

VI. Discussion and Conclusions

Diffusion of the multiplicative process of image formation leads to a power-law statistic in the Fourier domain. Where diffusion of additive fluctuations lead to a Boltzmann law, that is exponential decay of intensity over scale. Diffusion of unbounded multiplicative fluctuations lead to a log-normal correlation function. However, in the case of light reflectance from rough surfaces, the boundaries on the multiplicative process impose the constrained converging diffusion process to lead to a power-law correlation function over the diffusion scale [21], [8]. Hence, the scaling behavior of large image collections will conform to a power-law, imposed by the multiplicative nature of light reflectance and light observation, rather than by "self-similarity". Integration over numerous power-laws due to a spatial diffusion process yields a Weibull type distribution for filter responses. Hence, in general image collections, the response to any zero-average filter will follow the Weibull type of distributions. These trivial considerations predict the results as empirically obtained for large natural image collections [10], [11], [12], [13], [14].

Under these theoretical considerations, any geometrical entity derived from the long-range power-law correlation function are essentially derived from shadow and shading effects in the image. The distribution of contrast is indicative for the intrinsic size of texture details, as shading is by projection related to texture patches. Such size distributions may well be approximated by a sequential fragmentation process. Fragmentation is the continuous analog of branching random walks, a well known tool for texture analysis. In this respect, fragmentation theory may provide an additional tool in texture analysis and texture feature extraction.

The given analysis does not directly add to the current methodology in texture analysis, as power-laws are well known concepts, incorporated in numerous texture analysis or generation models. Rather, the importance is in the multiplicative processes, which can be shown to naturally result in power-law behavior. Furthermore, the given analysis tries to fit the multi-resolution concept of texture into the well founded and validated scale-space framework. Hence, the given analysis provides some new insight in texture analysis as function of observation scale. That is a new direction which may prove to be fruitfull in the near future.

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