Stochastic Mean Curvature Motion in Computer Vision: Stochastic Active Contours

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Abstract: *This paper presents a novel framework for image segmentation based on stochastic optimization. During the last few years, several segmentation methods have been proposed to integrate different information in a variational framework, where an objective function depending on both boundary information and region information is minimized using a gradient-descent method. Some recent methods are even able to extract the region model during the segmentation process itself. Yet, in complex cases, the objective function does not have any computable gradient. In other cases, the minimization process gets stuck in some local minimum, while no multi-resolution approach can be invoked. To deal with those two frequent problems, we propose a stochastic optimization approach and show that even a simple Simulated Annealing method is powerful enough in many cases. Based on recent work on Stochastic Partial Differential Equations (SPDEs), we propose a simple and well-founded method to implement the stochastic evolution of a curve in a Level Set framework. The performance of our method is demonstrated on both synthetic and real images.*

1. Introduction

This work is motivated by how stochastic motion can improve current shape optimization methods in Computer Vision. We are interested in a hypersurface evolution $\partial\Gamma(t)$, where $\Gamma(t)$ is a closed subset of $\mathbb{R}^N$ with non empty interior and $\partial\Gamma(t)$ evolves according to the equation

$$\frac{\partial(\partial\Gamma)}{\partial t} = (\kappa + \dot{W}(t,x))\mathbf{n} = \beta\mathbf{n} \tag{1}$$

where $\mathbf{n}$ is the normal to $\partial\Gamma(t)$ and where the normal velocity β depends on κ, the mean curvature of $\partial\Gamma(t)$, and $\dot{W}$, a stochastic perturbation, which will change $\Gamma(t)$ only through its normal component. The mean curvature motion $\beta = \kappa$ and its implementation with the Level Sets method [21, 25, 20] is well known. The novelty in this work is the implementation of the recently proposed stochastic curvature driven flows like equation (1) (see [28]) and its application to Computer Vision.

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Stochastic dynamics of interfaces have been widely discussed in later years in the physics literature. The work in fields like front propagation or front transition is aimed at modeling and studying the properties of a moving frontier between two media that is subject to macroscopic constraints and random perturbations (which are due to the bulk). The natural translation of this dynamic in mathematical language is done through Stochastic Partial Differential Equations (SPDEs). These equations were introduced by Walsh in [28] and their mathematical properties were studied using mostly partial differential equations tools. Nevertheless, the problems researchers have to deal with are various and there is more than one way to add a stochastic perturbation to a PDE. An up to date survey of the existing models on stochastic motions by mean curvature can be found in [30].

It was only in recent years that the notion of viscosity solution for a SPDE was developed by Lions and Souganidis in a series of articles [15, 16, 17, 18]. Their notion of weak viscosity solution is very attractive for numerical applications, since they define the solution as a limit in a convenient space for a set of approximations. Since their pioneering work, related work has been done by Yip [29] and by Katsoulakis et al [12]. Other approaches have also been proposed by Bally et al in [2], much in the spirit of Walsh [28], and recently by Buckdahn in a series of articles [5] which amounts to a different definition of viscosity solutions. Yet, we will make direct use of Lions and Souganidis results. Their extension of the viscosity solution notion is particularly adapted to the well known Level Sets method [20], making it even more interesting in the area of Computer Vision.

In the following, we will briefly present previous work aimed at modeling a surface that is subject to mean curvature motion coupled with noise perturbation. Afterward, we will discuss the implementation of such a dynamic in the Level Sets framework. Then we expose how this stochastic motion, coupled with Simulated Annealing [14], can be used in Computer Vision in the context of shape optimization problems. Finally, results are given in the particular case of active contours [24, 22, 6] demonstrating how the Active Contours can be improved in what could be called *Stochastic Active Contours*

2. Theoretical results

The mean curvature motion [8] is usually implemented with the Level Sets method [21, 25, 20]. The underlying mathematical tool is the theory of viscosity solutions for Partial Differential Equations (PDE).

Namely, let $u : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$ be the Level Sets function which describes, at time $t \geq 0$, the evolution of the domains $\Gamma(t)$:

$$\Gamma(t) = \{x \in \mathbb{R}^N \,:\, u(t,x) \leq 0\}$$
$$\partial\Gamma(t) = \{x \in \mathbb{R}^N \,:\, u(t,x) = 0\}$$

It is well known that making $\partial\Gamma(t)$ move with normal velocity equals to its curvature $\beta = \kappa$ amounts to u satisfying the Partial Differential Equation (PDE)

$$\frac{\partial u}{\partial t} = |\mathrm{D}\, u| \operatorname{div}\left(\frac{\mathrm{D}\, u}{|\mathrm{D}\, u|}\right) \tag{2}$$

This equation admits unique globally defined uniformly continuous solutions in the viscosity sense (see [21, 25, 20]).

2.1. Viscosity solutions

Recently, the stochastic dynamics given by equation (1), ie. $\beta = \kappa + \dot{W}(t,x)$, have been addressed in [16] and a notion of weak solution has been developed for this type of equation. Equation (1) is in fact a shortcut for:

$$\mathrm{d}(\partial\Gamma) = \kappa\, \mathbf{n}\, \mathrm{d}\, t + \mathbf{n} \circ \mathrm{d}\, W \tag{3}$$

(see Walsh [28]), where the symbol $\circ\, \mathrm{d}\, W$ stands for the Stratonovich integral, which, unlike the Itô integral, is well suited for stochastic geometry, since it does not change when the coordinates change (see [7, 10] for an introduction). Lions and Souganidis state that the corresponding SPDE in the Level Sets framework, namely

$$\mathrm{d}\, u = |\mathrm{D}\, u| \operatorname{div}\left(\frac{\mathrm{D}\, u}{|\mathrm{D}\, u|}\right) dt + |\mathrm{D}\, u| \circ \mathrm{d}\, W(t,x), \tag{4}$$

admits a unique solution in some viscosity sense. The uniqueness of the solution in the viscosity sense is of great importance, since it ensures that at all times the zero level set $\{x \in \mathbb{R}^N \,:\, u(t,x) = 0\}$ does not develop a non empty interior - phenomenon also known as *fattening*.

Due to lack of place, we will not expose the complete theory. Briefly, Lions and Souganidis define the viscosity solutions of

$$\begin{cases} u_t = F(\mathrm{D}^2\, u, \mathrm{D}\, u) + \sum_{i=1}^{M} H_i(\mathrm{D}\, u) \circ \mathrm{d}\, W(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

where F is continuous and degenerate elliptic and H is continuous and positively homogenous of degree one (for more details, see [15])

By replacing the Gaussian noise with finite variation approximations, they obtain a class of approximations

$$\begin{cases} u_t^\epsilon = F(\mathrm{D}^2\, u^\epsilon,\, \mathrm{D}\, u^\epsilon) + \sum_{i=1}^{M} H_i(\mathrm{D}\, u^\epsilon)\dot{\xi}_i^\epsilon(t,\, x) \\ u^\epsilon = u_0^\epsilon \end{cases} \tag{5}$$

for which convergence when $\epsilon \to 0$ is proved using mainly the method of characteristics in PDEs. Consequently, their result *allows us to simulate the solutions of such equations and be sure that the result of our computer simulation is what we expect it to be*.

Further more, we mention that according to Lions, the convergence takes place in $C(\mathbb{R}_+ \times \mathbb{R}^N)$, which means that the numerical solutions we develop will be continuous and that they will be converge uniformly almost surely in $\omega \in \Omega$, the space of the possible realizations.

2.2. Noise

One aspect of the equation (4) that we have not covered so far is the noise introduced in the equation. In the sequel, we will introduce briefly the stochastic Brownian sheet and try to describe some of its immediate properties and consequences on our equation. For a basic introduction on stochastic processes, see [10].

To add noise to a PDE, one would typically add a Gaussian to the stepping scheme made up for the equation. This gives rise to independent increments, both in time and space. Hence the idea of Brownian sheet. The same intuition resides in a very nice example given by Walsh in [28].

Formally, the Brownian sheet is a process defined as

$$W : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}$$

where Ω is a space of labels of realizations. Each $W(\omega,t,x) = W(t,x)$ is thus a real-valued Gaussian random variable with mean zero and variance $\langle W(t,x)\rangle = tx_1 \ldots x_N$.

The definition of an integral with respect to this process takes the same path as the definition of the classical stochastic integral. For details, see for example [28].

The usage of the stochastic integral always gives rise to a quadratic variation term, which has to be controlled. This is not always obvious in a classical framework when using the Brownian sheet. Therefore, we mention the work of [12], who study the same type of equation. Their solution in order to control the explosive character of the stochastic term, is to consider a *colored white noise* term. This corresponds to limiting the numbers of independent sources of noise and thus allows to having a more regular behavior.

3. Implementation

We now focus on practical ways of implementing the stochastic evolution given by equation (4).

As a first step, we use the following explicit first order scheme:

$$u(t+\Delta t) = u(t) + |\mathrm{D}\,u|\,\mathrm{div}(\frac{\mathrm{D}\,u}{|\mathrm{D}\,u|})\Delta t \\ + |\mathrm{D}\,u|\,(W_x(t+\Delta t) - W_x(t))$$

where $W_x(t+\Delta t) - W_x(t) \sim W_x(\Delta t) \sim \sqrt{\Delta t}\mathcal{N}(0,1)$, since $W_x(t)$ is a standard Brownian motion. Hence, we implement

$$u(t+\Delta t) = u(t) \\ + |\mathrm{D}\,u| \left[\mathrm{div}(\frac{\mathrm{D}\,u}{|\mathrm{D}\,u|})\Delta t + \mathcal{N}(0,1)\sqrt{\Delta t}\right]$$

using a standard narrow-banded procedure like the one described in [23].

Using the results mentioned in the last section, we claim that the above algorithm will converge, when $\Delta t \to 0$, towards the solution of equation (4).

Note that $\dot{W}$ is computed only on the space grid of the Level Sets implementation we are considering. As mentioned earlier, there are problems when considering a Brownian sheet due to the explosive character of the quadratic variation term. Nevertheless, we did not deal with this problem, since for the applications we developed we have a lower bound on the space grid dimension, which is given by the resolution of the underlying image. Thus, the noise we consider is strongly correlated in the x variable. Moreover, if some more spatial regularity in noise is needed, the noise can be computed on a coarser grid and interpolated in the near grid points. Figure 1 illustrates the effect of smoothing $\dot{W}$ in space: a more spatially smooth noise, gives more regular but larger oscillations of the surface. Note also that the curvature has a strong role with respect to this aspect, since it stops the contour to bend excessively. In doing this, the contour preserves nice properties such as the short-time connectivity. To emphasize this idea, we mention that in the absence of the curvature term (and thus being exposed to a completely random dynamic), the contour tends to break up around the main line and to develop bubbles. Despite the previous theoretical results, the properties of the stochastic mean curvature motion are still unknown, as were the properties of the classical mean curvature motion in the pioneering work by [8].

4. Applications to Computer Vision

Many Computer Vision problems consist in recovering a certain surface or region through a shape optimization framework [6, 22, 9]. The dynamics presented earlier, coupled with a decision mechanism, can be used to select such regions. This is why another ingredient we turned our attention to is the Simulated Annealing algorithm.

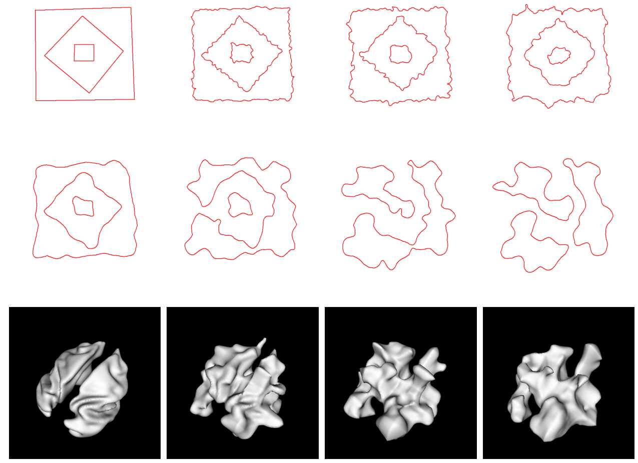

Figure 1: Different Stochastic Mean Curvature motions. Top row: starting from the initial curve (top left), three time steps of the evolution with Gaussian noise. Middle row: from the same initial curve, four time steps of the evolution with a spatially smoother noise. Bottom row: a 3D example starting from the cortex of a monkey.

Based on the work of Metropolis et al., Simulated Annealing was first mentioned by Kirkpatrick in [14] as a nice application of statistical physics to optimization problems. Its purpose is to introduce a probabilistic decision mechanism for finding global minima in higher dimension.

As it will be seen further, the combination of the stochastic mean curvature dynamics with this selection algorithm can be a powerful tool in Computer Vision, for instance in the context of active contours [6]. As opposed to the dynamics introduced earlier, the use of Simulated Annealing in the area of active contours is not a complete novelty. We would like to briefly comment upon the previous works oriented toward the use of genetic programming in this field.

4.1. Comparison with previous work in Computer Vision

In a lot of cases, the stochastic theory is used to help researchers develop an intuition of the macroscopic dynamics at a microscopic level. This if, for instance, the case in [3], where an algorithm for stochastic approximations to a curve shortening flow are built. Another example is given by [27], where the authors develop a model of anisotropic diffusion using the information gained by analyzing the stochastic differential equation associated to a linearized version of the geometric heat equation.

In other cases, stochastics are actively used in selection algorithms meant to overcome some classical dynamics difficulties. In [26] Storvik used Simulated Annealing combined with a Bayesian dynamics and developed applications in medical imagery. He used a node-oriented representation technique for the contour representation. Thus, his algorithm can only detect simply connected domains in an image. Moreover, self-intersections are not allowed, due to the complications they would involve. The Bayesian techniques used for his contour evolution were therefore highly limited (perhaps due to reduced computing power available at the time) and the applications presented only make use of 3 pixels being changed in a time step.

More recently, Ballerini et al developed in [1] an interesting application to medical image segmentation using a genetic algorithm, *genetic snakes*. They used a model that they fit using a number of control points. Their application cannot, therefore, be extended to a more general framework.

In conclusion, it is important to notice that the main ingredient of our work is not the Simulated Annealing part, but rather the underlying dynamics presented earlier. It is obvious that the stochastic approach adds to the power and flexibility of the Level Sets technique into a very powerful tool. We can thus use this mechanism through skillfully applied controls, while continuing to allow for topological changes and weak regularity assumptions. Moreover, the presence of the stochastic terms tends to help the dynamics grow towards non convex shapes, which was another drawback of the classical method.

Simulated Annealing is used in our experiments. In the future, more evolved genetic programming selection techniques might be considered, but it is encouraging that such simple ingredients added to the Level Sets framework provide good practical results. *Sketchily, one can see the same difference between our method and the previous ones, than between geodesic active contours and the pioneering snakes [11].*

4.2. Principle

Given some Computer Vision problem in a variational framework where we have to find the region Γ that minimizes an energy $E(\Gamma) = E(u)$, we use the following simple Simulated Annealing decision scheme:

1. Start from some initial guess u_0
2. compute u_{n+1} from u_n using the dynamics presented earlier
3. compute the energy $E(u_{n+1})$
4. accept u_{n+1}:
   - if $E(u_{n+1}) < E(u_n)$
   - otherwise, accept u_{n+1} with probability $\exp\left(-\frac{E(u_{n+1})-E(u_n)}{T(n)}\right)$
5. loop back to step 2, until some stopping condition is fulfilled

Here, $T(n)$ is a time-dependent function that plays the same role as a decreasing temperature. Its choice is not obvious. If the temperature decreases to fast, the process may get stuck in a local minimum; on the contrary, decreasing too slowly in order to reach the global minimum may be computationally expensive. A classical choice is $T(n) = T_0/\sqrt{n}$.

4.3. Remarks and motivation

The classical way to solve the previous minimization problem is often to use a gradient descent method. The Euler-Lagrange equation is computed, leading to some PDE $\partial\Gamma_t = \beta_c \mathbf{n}$. In that case, we use the classical motion as heuristics that drive the evolution faster toward a minimum, and replace the dynamics of step 2, by

$$\beta = \beta_c + \kappa + \dot{W}(t,x)$$

or even by $\beta = \beta_c + \dot{W}(t,x)$ when β_c already contains a curvature term.

As often with genetic algorithms, the proof of the convergence of this algorithm toward a global minimum is still an open problem. However, practical simulations indicate that the above algorithm is more likely to overcome local minima than the classical approach. This is our main motivation, since local minima are the major problem of classical approaches. Note also that our framework can be used in cases when the Euler-Lagrange equation is too complex from a mathematical or computational point of view, or even impossible to compute.

5. Stochastic Active Contours

Our scheme could be used in the Geodesic Active Contours framework [6] where segmentation is based upon gradient intensity variations. Yet, a multiscale approach is often used successfully in that context to overcome the local minimum problem. However, many other segmentation schemes [22] use a region model (eg. texture, statistics) often unadapted to multiscale or unusable at coarse scales. We will first focus on one such case, namely a single Gaussian statistics model by Deriche and Rousson [24].

5.1. Single Gaussian model

In their active and adaptive segmentation framework [24], the authors model each region of a gray-valued of color image I by a single Gaussian distribution of unknown mean μ_i and variance Σ_i. The case of two regions segmentation turn into minimizing the following energy:

$$E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) = \int_\Gamma e_1(x) + \int_{D/\Gamma} e_2(x) + \nu \mathrm{length}(\partial\Gamma)$$

with

$$e_i(x) = -\log p_{\mu_i \Sigma_i}(I(x))$$

where

$$p_{\mu_i \Sigma_i}(I(x)) = C|\Sigma_i|^{-1/2} e^{-(I(x)-\mu_i)^T \Sigma_i^{-1}(I(x)-\mu_i)/2}$$

is the conditional probability density function of a given value $I(x)$ with respect to the hypothesis (μ_i, Σ_i). The parameters (μ_i, Σ_i) depending on Γ, the energy is actually a function of Γ only: $E(\Gamma, \mu_1, \Sigma_1, \mu_2, \Sigma_2) = E(\Gamma)$. Its Euler-Lagrange equation is not obvious, but finally simplifies into the minimization dynamics

$$\beta_c = e_2(x) - e_1(x) + \nu \operatorname{div}\left(\frac{\mathrm{D}\, u}{|\mathrm{D}\, u|}\right)$$

The authors successfully segment two regions even with same mean. However, the evolution could easily be stuck into some local minimum and a multiscale approach might modify the statistics of the region so that no segmentation would be possible anymore. As demonstrated figure 2, a simple Simulated Annealing scheme with dynamics $\beta = \beta_c + \dot{W}(t, x)$ overcomes this problem. Figure 3 shows the same phenomenon on a real image. Note that this image was successfully segmented by Paragios and Deriche with their active region framework [22]. Yet, they used an adapted model of texture. Here, the Stochastic Active Contours framework succeeds in making a simple single Gaussian model with unknown parameters find the correct regions.

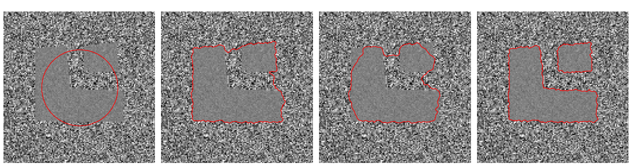

Figure 2: Segmentation of two regions modeled by two unknown Gaussian distributions (same mean, different variances). From left to right: (i) the initial curve, (ii) the final state of the classical approach [24] stuck in a local minimum, (iii) and (iv) an intermediate and the final state of our method

5.2. Gaussian mixtures

As an illustration of the case when the Euler-Lagrange equation cannot be computed, we simply extend the previous model to region statistics modeled by a mixture of Gaussian distributions of parameters $\Theta_i = (\pi_i^1, \mu_i^1, \Sigma_i^1, ..., \pi_i^{n_i}, \mu_i^{n_i}, \Sigma_i^{n_i})$. with $\sum_j \pi_i^j = 1$. The conditional probability density function of a given value $I(x)$

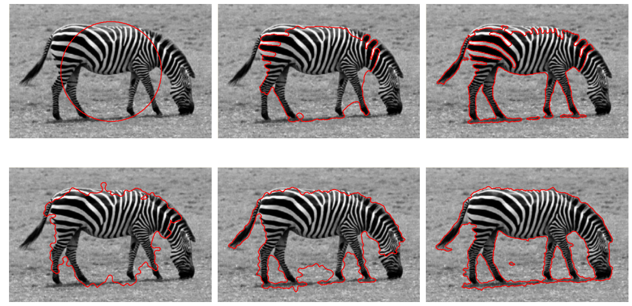

Figure 3: Segmentation of two regions modeled by two unknown Gaussian distributions. Top row: the initial curve, an intermediate and the final time step of the classical method, again stuck in a local minimum. Bottom row: two intermediate steps and the final step of our method.

becomes:

$$p_{\Theta_i}(I(x)) = \sum_{j=1}^{n_i} \pi_j p_{\mu_i^j \Sigma_i^j}(I(x))$$

The number of Gaussian distributions can be given, estimated at initial time step, or dynamically evaluated. A large literature is dedicated to the problem of estimating Θ_i from input samples. We use here the original k-means algorithm pioneered by MacQueen [19], although we have tested extensions like the fuzzy k-means [4].

Our segmentation problem still consists in minimizing the same energy, with now $e_i(x) = -\log p_{\Theta_i}(I(x))$. Unfortunately, we now have to deal with a complex dependency of Θ_i with respect to Γ. In fact, the k-means algorithm acts as a "black box" implementing $\Gamma \rightarrow \Theta_i(\Gamma)$. As a consequence, the Euler-Lagrange equation of the energy $E(\Gamma, \Theta_1(\Gamma), \Theta_2(\Gamma)) = E(\Gamma)$ cannot be computed. A deterministic contour evolution driven by $\beta_c = e_2 - e_1 + \nu\kappa$ does not always converge, even to a local minimum, due to the fact that $\beta_c \mathbf{n}$ is not the exact gradient (see figure 4). Yet, the Stochastic Active Contours can still be used, with β_c as heuristics (figure 4 again).

Even when the deterministic scheme converge more or less, our method shows a better ability to overcome local minima: figure 5 illustrated the case of a standard geometric energy barrier caused by narrow pathways while figure 6 shows how Θ_i can be stuck leading to a dramatic evolution toward completely false regions (see also attached multimedia material). Finally, figure 7 shows some more examples on other real images. Animations corresponding to all the presented examples can be downloaded at `http://cermics.enpc.fr/˜juan/SAC/`.

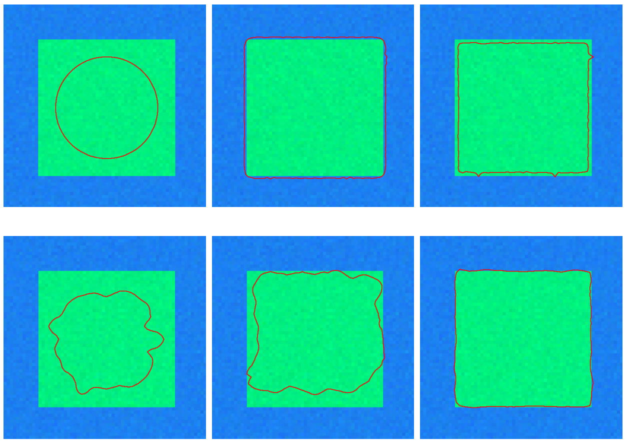

Figure 4: Segmentation of two regions modeled by two unknown Gaussian mixtures (here, only one Gaussian by mixture!) Top row: the initial curve, and two states of the deterministic method, states between which the final curve oscillates, a behavior caused by an incorrect gradient. Bottom row: two intermediate steps and the final step of our method, using the same gradient as heuristics.

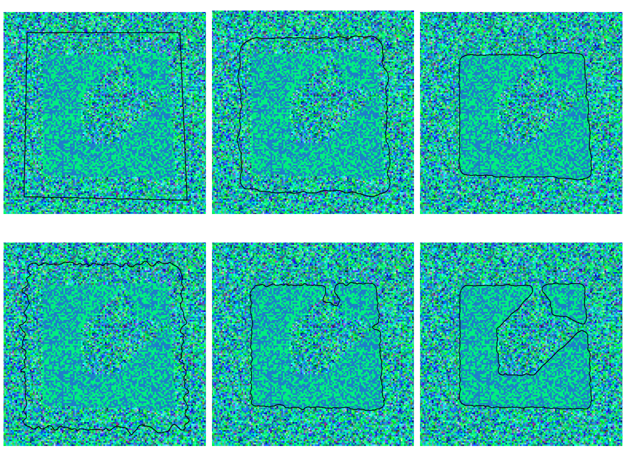

Figure 5: Segmentation of two regions modeled by two unknown Gaussian mixtures (here, two identical mixtures of two colors with only different variances). Top row: the initial curve, an intermediate and the final time step of the deterministic method, again stuck in a local minimum. Bottom row: two intermediate steps and the final step of our method.

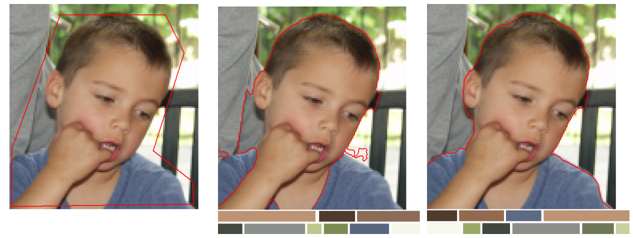

Figure 6: Segmentation of two regions modeled by two unknown Gaussian mixtures. From left to right: (i) The initial curve, (ii) the final state of the deterministic method, stuck in a local minimum and (iii) the final state of our method. The two lines of colored rectangles below the images indicate the means of the mixtures components and their respective weights (Top line for the inside region, bottom line for the outside region)

6. Conclusion

Based on recent work on Stochastic Partial Differential Equations, we have presented propose a simple and well-founded method to implement the stochastic mean curvature motion of a surface in a Level Set framework. This method is used as the key point of a stochastic extension to standard shape optimization methods in Computer Vision. In the particular case of segmentation, we introduced the *Stochastic Active Contours*, a natural extension of the well-known active contours. Our method overcome the local minima problem and can also be used when the Euler-Lagrange equation of the energy is out of reach. This extension is not time consuming: the only computational effort is computing the energy, which can generally be done by a simple run through the domain of level set function. Convincing results are presented with the segmentation of regions modeled by unknown statistics, namely single Gaussian distributions or mixtures of Gaussian distributions. The way is now open for applying our principle to other Computer Vision problems but also in different fields where shape optimization problems arise, like in theoretical chemistry [13].

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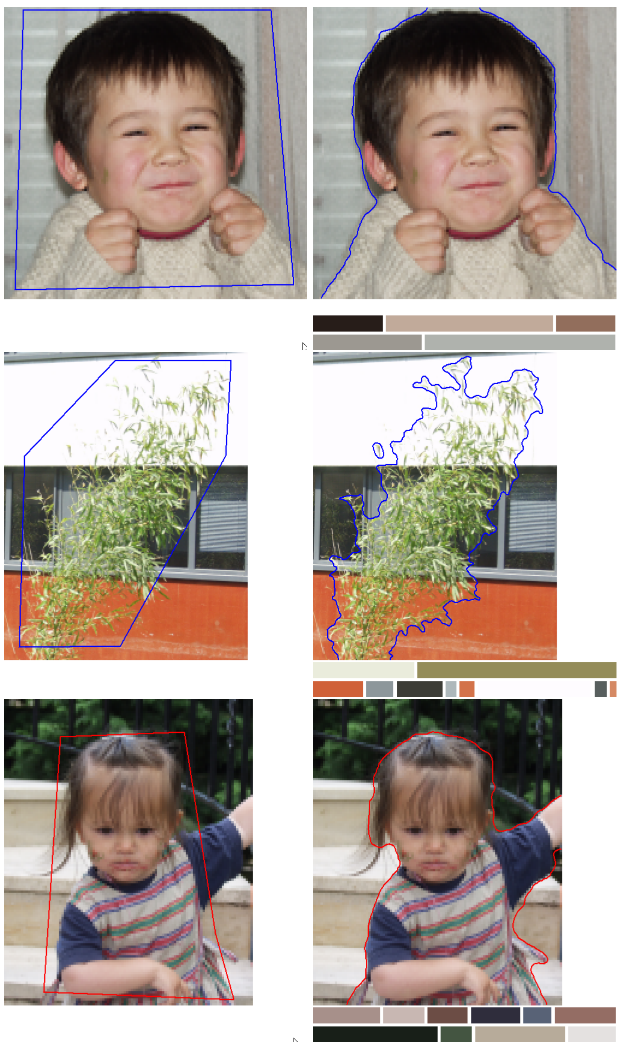

Figure 7: Segmentation of two regions modeled by two unknown Gaussian mixtures. Left column: the initial states. Right column: the corresponding final states of our method.

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